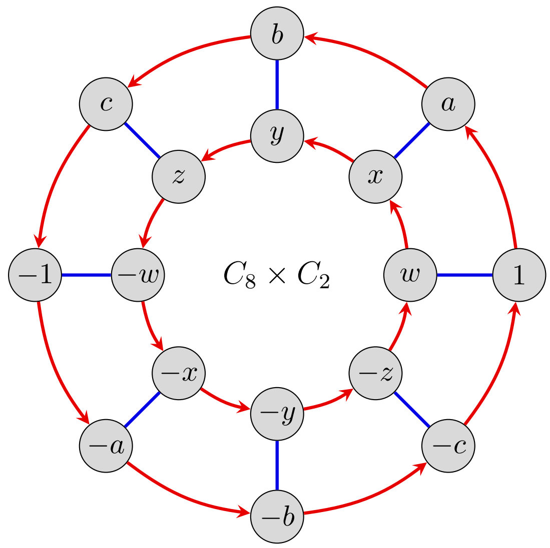
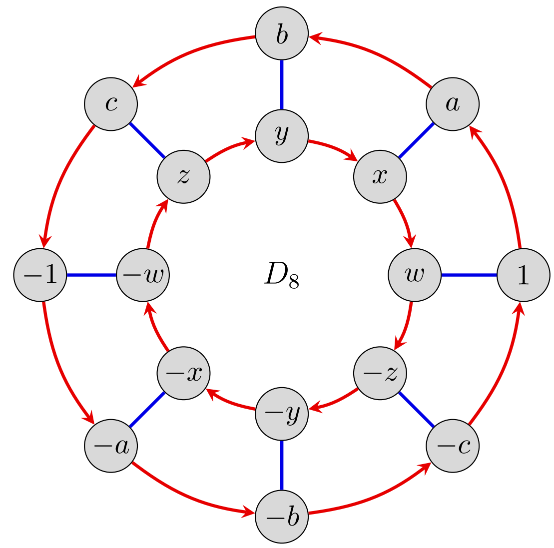
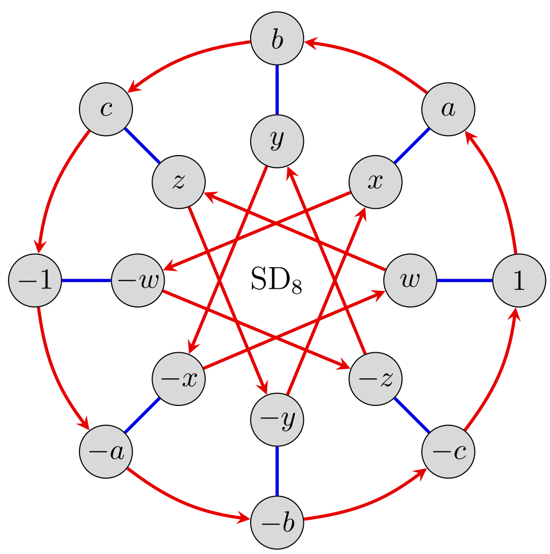
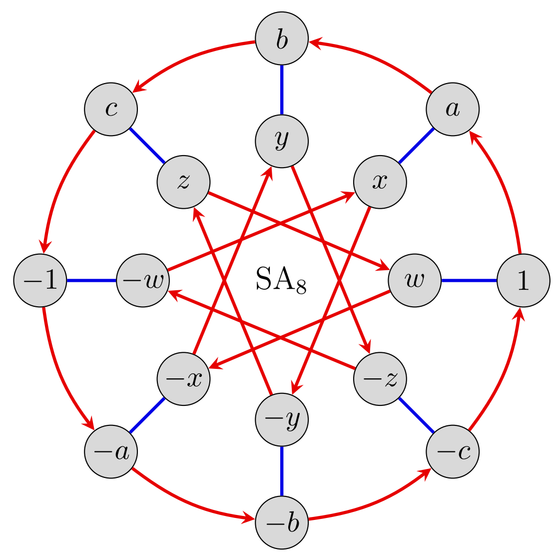
1. Cayley diagrams for the four semidirect products of $C_8$ with $C_2$ are shown below, with a different labeling scheme on their nodes.




For all four of these groups, identifying each element with its "negative" yields a "quotient group" of order 8, like what we did with the dicyclic group $\mathrm{Dic}_8 = Q_{16}$ in HW 2. Construct a Cayley table and Cayley diagram for each of these quotient groups, using the elements

$$\pm 1, \quad \pm a, \quad \pm b, \quad \pm c, \quad \pm w, \quad \pm x, \quad \pm y, \quad \pm z,$$

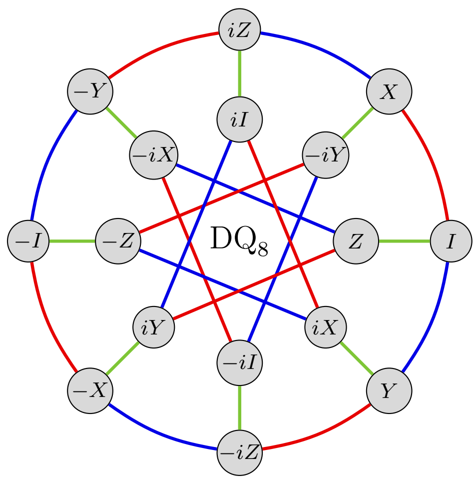
and determine to which familiar group each is isomorphic. If two groups give the same table and diagram, you do not need to write this out twice.

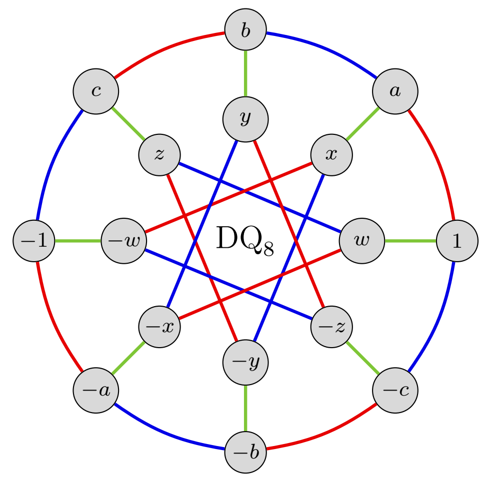
2. The *diquaternion group* $\mathrm{DQ}_8$ can be constructed from our standard representation of $Q_8 = \langle R_4, S, T\rangle$, along with the reflection matrix $F$ in $D_n = \langle R_n, F\rangle$. That is,

$$\mathrm{DQ}_8 \cong \langle i, j, k, f\rangle \cong \left\langle \underbrace{\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}}_{R=R_4}, \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{S}, \underbrace{\begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}}_{T=RS}, \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{F} \right\rangle.$$

A Cayley diagram for $\mathrm{DQ}_8$ generated by the *Pauli matrices* $X, Y, Z$, is shown below (left).

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$




(a) Find a presentation for $\mathrm{DQ}_8 = \langle X, Y, Z\rangle$.

(b) Construct a Cayley diagram for $\mathrm{DQ}_8 = \langle R, S, F\rangle$, and find a group presentation. Extra credit will be given for the best-looking construction.

(c) Carry out the "quotient process" as was done for $C_8 \times C_2$, $D_8$, $\mathrm{SD}_8$, and $\mathrm{SA}_8$ in the previous problem; see the diagram on the right.

3. For this problem, the use of an online matrix calculator, like `https://matrixcalc.org`, that can handle complex exponential inputs, is strongly recommended.

(a) The dicyclic group $\mathrm{Dic}_n$ is only defined when $n$ is even. However, if we try to define

$$\mathrm{Dic}_3 := \left\langle \begin{bmatrix} \zeta_3 & 0 \\ 0 & \bar{\zeta}_3 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\rangle,$$

then the result is still a group. Determine which group this is, with justification.

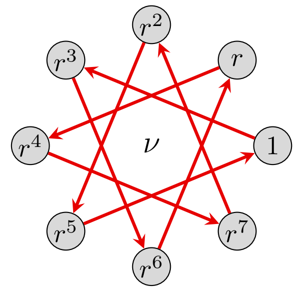
(b) The diquaternion and semidihedral groups, $\mathrm{DQ}_n$ and $\mathrm{SD}_n$, are only defined when $n = 2^m$. However, we can still define

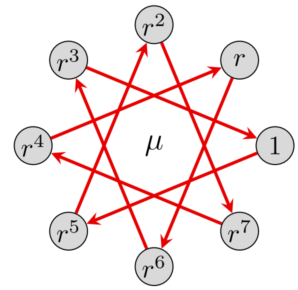
$$\mathrm{DQ}_6 := \left\langle \begin{bmatrix} \zeta_6 & 0 \\ 0 & \bar{\zeta}_6 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\rangle, \qquad \mathrm{SD}_3 := \left\langle \begin{bmatrix} \zeta_3 & 0 \\ 0 & -\bar{\zeta}_3 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\rangle.$$

What groups are these? Construct a Cayley diagram for each.

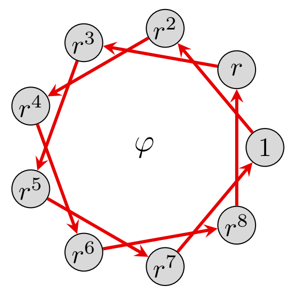
4. The automorphism group of $C_n$ is isomorphic to $U_n$, the multiplicative group of integers modulo $n$, from HW 2, #2. For each of the following, construct a Cayley diagram of $\mathrm{Aut}(C_n)$ with the nodes labeled by re-wirings, and a Cayley table for this group. Then determine its isomorphism type.

   (a) $\mathrm{Aut}(C_8) = \langle \nu, \mu \rangle$, defined by

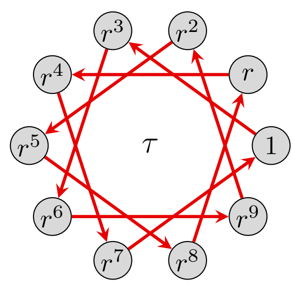



   (b) $\mathrm{Aut}(C_9) = \langle \varphi \rangle$, defined by




   (c) $\mathrm{Aut}(C_{10}) = \langle \tau \rangle$, defined by




   (d) $\mathrm{Aut}(C_{16}) = \langle \rho, \sigma \rangle$, defined by