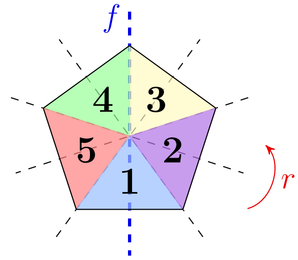
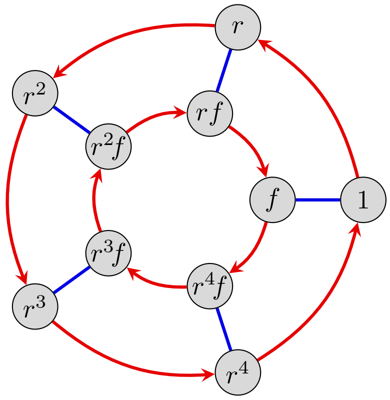
1. All of the subgroups of D_5 should be visually apparent from thinking about symmetries of a regular pentagon, shown below at left. At right is a Cayley graph.





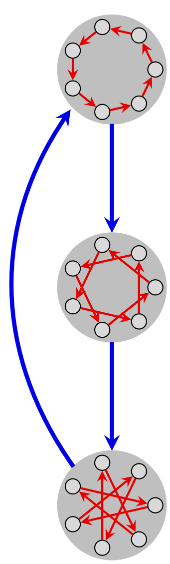
   (a) Construct a subgroup lattice for D_5. Label each edge from H to K with $[H : K]$.

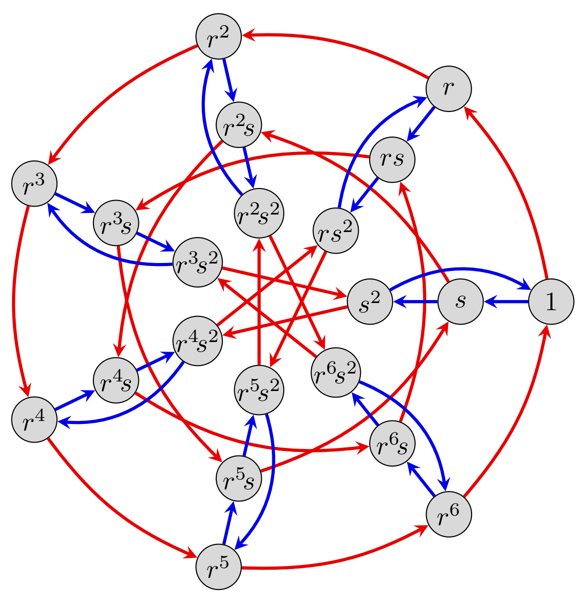
   (b) Find the left and right cosets of the subgroups $\langle r \rangle$ and $\langle f \rangle$.

   (c) The *normalizer* of $H \leq G$, denoted $N_G(H)$, is the union of the left cosets of H that are also right cosets. Find the normalizer of $\langle r \rangle$ and $\langle f \rangle$.

   (d) Two subgroups $H, K \leq G$ are *conjugate* if $K = gHg^{-1} := \{ghg^{-1} \mid h \in H\}$ for some $g \in G$. This defines an equivalence relation on the set of subgroups called *conjugacy classes*. Partition the subgroups of D_5 into conjugacy classes.

2. Cayley graph of the smallest non-abelian group of odd order, $G = C_7 \rtimes C_3$, is shown below, highlighting its semidirect product structure.





   (a) On a blank Cayley graph, label nodes with the order of the corresponding elements. Then construct a cycle graph, labeled by group elements.

   (b) Construct a subgroup lattice and label each edge with the corresponding index.

   (c) Find the left and right cosets of the subgroups $\langle r \rangle$ and $\langle s \rangle$, and their normalizers.

   (d) Partition the subgroups into conjugacy classes, and denote this on your lattice.

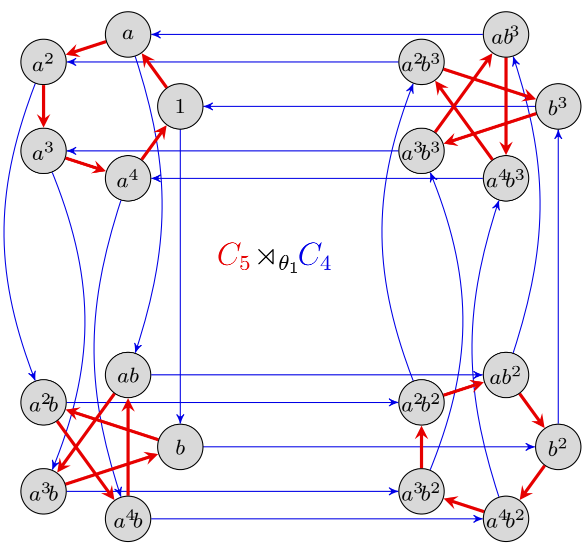
3. In this problem, you will construct the semidirect product $C_9 \rtimes C_3$. Recall that $\mathrm{Aut}(C_9)$ was constructed on the previous assignment.

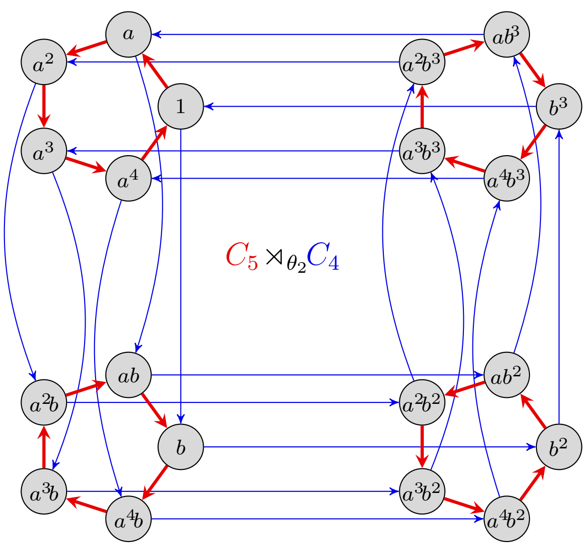
   (a) Find all possible labeling maps $\theta\colon C_3 \to \mathrm{Aut}(C_9)$.

   (b) Construct a nonabelian semidirect product of $C_9 = \langle r \rangle$ with $C_3 = \langle s \rangle$, using a labeling map that makes the Cayley graph less tangled. Include a Cayley graph of C_3 with the nodes labeled by $\theta(s^j)$, and a Cayley graph of $C_9 \rtimes_\theta C_3$, with the nodes labeled by $r^i s^j$.

   (c) Repeat Problem 2, but for the group $G = C_9 \rtimes C_3$. It is helpful to know that it has four subgroups of order 9 and four subgroups of order 3.

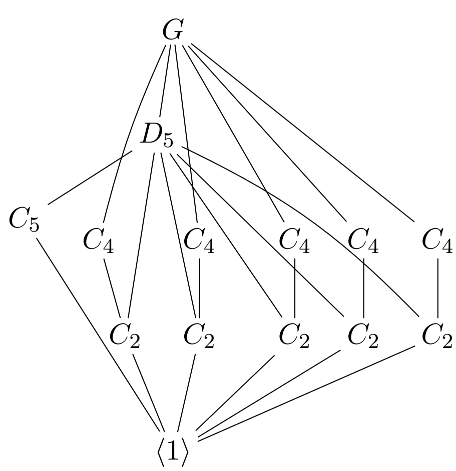
4. Consider two semidirect products of C_5 with C_4, whose Cayley graphs are shown below.

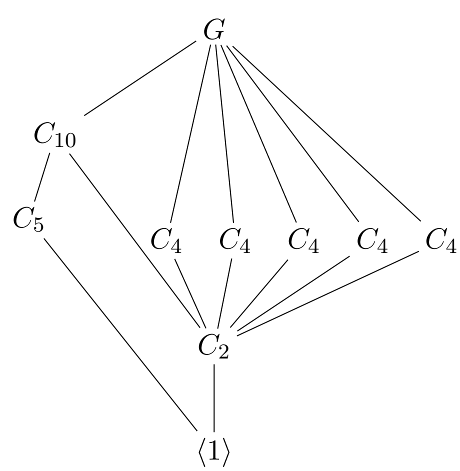




   (a) On blank Cayley graphs, label the order of each element. Then construct a cycle graph, with the nodes labeled by group elements.

   (b) The subgroup lattices of these two groups are shown below, not necessarily in the right order. Determine which lattice corresponds to which Cayley graph (with justification), and then re-draw them with the subgroups written by generators.





(c) Determine which group each of these is isomorphic to, and which elements a and b correspond to. Recall that there are only three nonabelian groups of order 20:

$$D_{10} = \langle r, f \mid r^{10} = f^2 = 1,\, rfr = f \rangle, \qquad \mathrm{Dic}_{10} = \langle r, s \mid r^{10} = s^4 = 1,\, r^5 = s^2 \rangle,$$

$$\mathrm{AGL}_1(\mathbb{Z}_5) = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle \leq \mathrm{GL}_2(\mathbb{Z}_5).$$

Write a presentation for both groups in this problem, in terms of a and b.

(d) Construct the subgroup lattice for $G = D_{10}$. It helps to think of the subgroups geometrically – there are two subgroups isomorphic to D_5, unique cyclic subgroups of orders 10 and 5, five subgroups isomorphic to V_4, and 11 subgroups of order 2.

(e) For each of the diagrams below, determine whether it is the Cayley graph of a group. If yes, write a presentation and determine whether it is isomorphic to D_{10}, Dic_{10}, or $\mathrm{AGL}_1(\mathbb{Z}_5)$. If no, explain why.