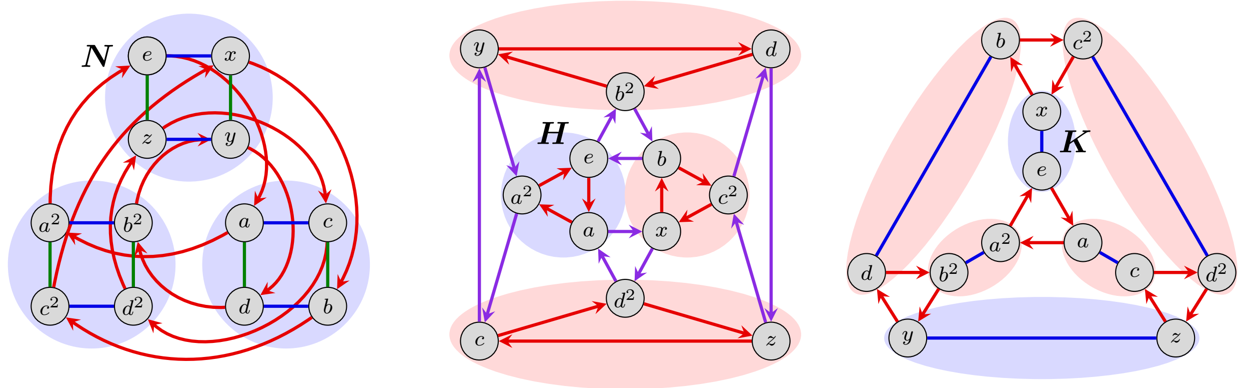
1. Below are three Cayley diagrams of $A_4$, each highlighting the left cosets of a different subgroup. We can take $a = (123)$, $b = (134)$, $x = (12)(34)$, and $z = (13)(24)$.




   (a) For each subgroup shown above, partition $A_4$ into its right cosets. Write them as subsets of $A_4$, consisting of permutations in cycle notation. Also, highlight them by colors on a fresh copy of the Cayley diagrams.

   (b) For each left coset $gH$, illustrate the construction of the conjugate subgroup $gHg^{-1}$ on a fresh copy of the Cayley diagram. Repeat this for $N$ and $K$.

2. Show that $A \times \{1_B\}$ is a normal subgroup of $A \times B$, where $1_B$ is the identity element of $B$. That is, show first that it is a subgroup, and then that it is normal.

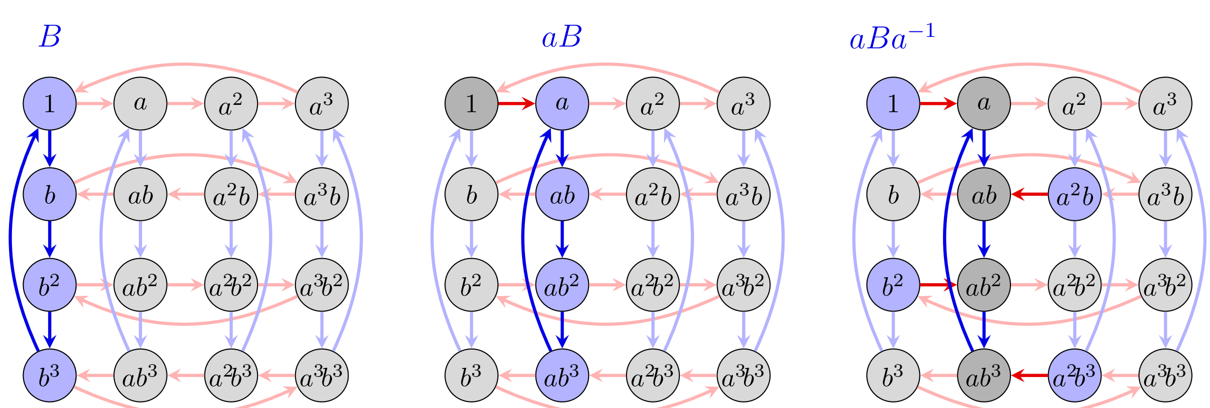
3. Let $G$ be a group, not necessarily finite, and $H \leq G$.

   (a) Show that for any fixed $x \in G$, we have an equality $\{gx \mid g \in G\} = G$ of sets.

   (b) Show that the subgroup
$$N := \bigcap_{x \in G} xHx^{-1}$$
   is normal in $G$.

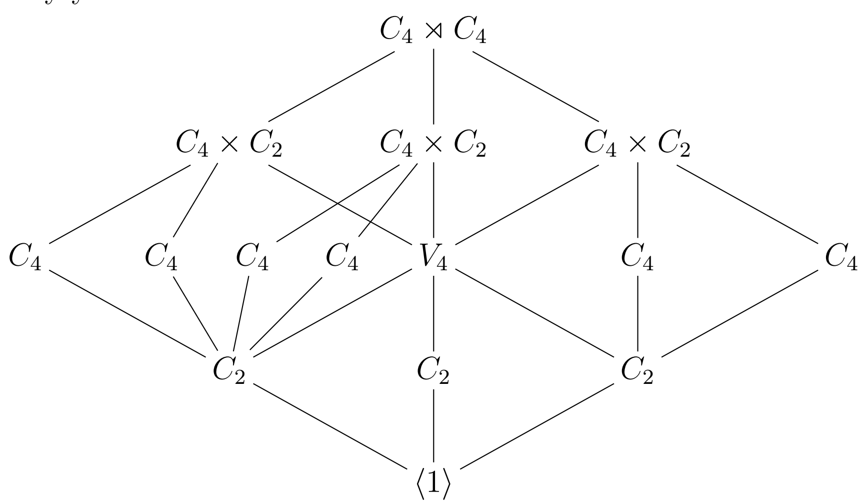
   (c) Show that *every* normal subgroup $K \trianglelefteq G$ contained in $H$ is contained in $N$. In other words, $N$ is the *largest normal subgroup* of $G$ contained in $H$.

4. Shown below is a Cayley diagram for $G = C_4 \rtimes C_4 = \langle a, b \rangle$, and the construction of the conjugate subgroup $aBa^{-1}$, where $B = \langle b \rangle$.



(a) For both order-4 subgroups, $\langle ab\rangle$ and $\langle a^2, b^2\rangle$, illustrate the construction of its conjugate subgroups on the Cayley diagram, in the same 3-step process that was done above for $B = \langle b\rangle$. Carry this out for each of its three distinct left cosets (excluding the subgroup itself).

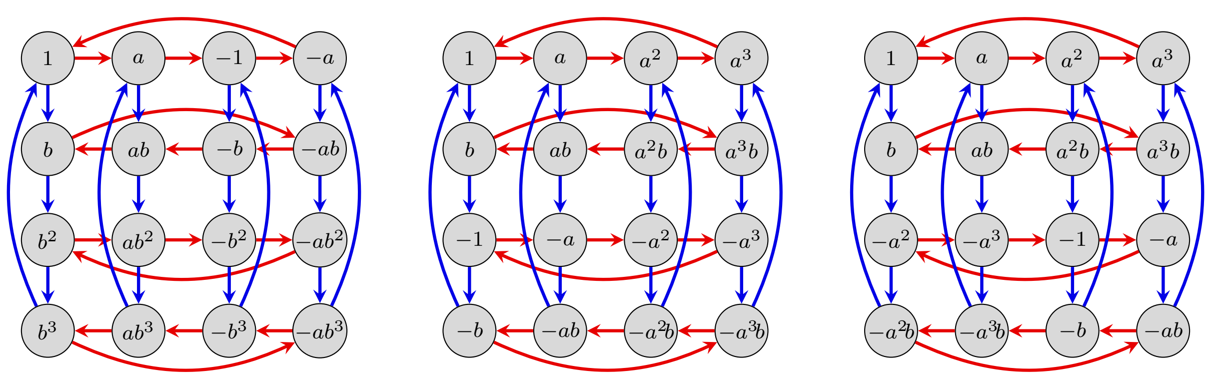
(b) The subgroup lattice of $G = \langle a, b\rangle$ is shown below. Re-draw this with the subgroups written by generator(s). Then partition the subgroups into conjugacy classes, and fully justify your answer.




(c) Without computing left or right cosets, i.e., only from the knowledge of conjugacy classes, find the normalizer of each subgroup. Justify your answer.

(d) Construct a (labeled) cycle diagram. Which subgroup is the center, $Z(G)$?

(e) Shown below are three copies of the Cayley diagram for $G$, each with a peculiar labeling of the nodes. In each diagram, a pair of nodes that differ by a sign represent a left coset of one of the three order-2 subgroups, all of which are normal.




For each diagram, construct a Cayley table and Cayley diagram consisting of these eight "cosets". Determine what the resulting *quotient group* is isomorphic to, and describe where that subgroup lattice appears, hiding in the lattice for $G$.

5. Let $H$ be a subgroup of $G$. Given two fixed elements $a, b \in G$, define the sets

$$aHbH = \{ah_1bh_2 \mid h_1, h_2 \in H\} \qquad \text{and} \qquad abH = \{abh \mid h \in H\}.$$

Show that if $H \trianglelefteq G$, then $aHbH = abH$.