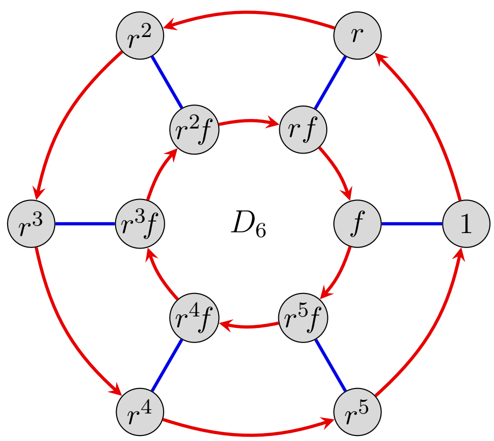
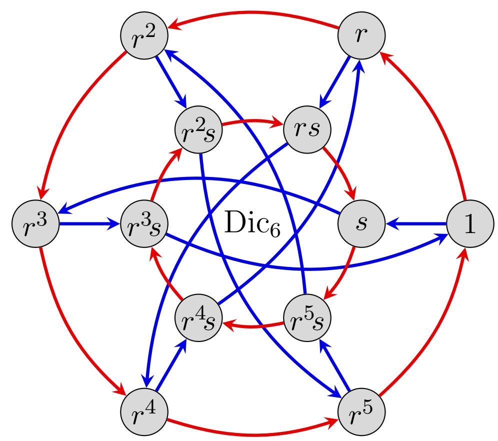
1. In both the dihedral group $G = D_6$ and dicyclic group $G = \mathrm{Dic}_6$, whose Cayley graphs are shown below, the subgroups $N = \langle r^3 \rangle$ and $H = \langle r^2 \rangle$ are normal. For both, construct a Cayley table and Cayley graph for the quotients G/N and G/H, and determine what these are isomorphic to.





2. Let H be a subgroup of G.
   (a) Show that if G is abelian, then H and G/H are abelian.
   (b) Show that if $G/Z(G)$ is cyclic, then G is abelian.
   (c) What cyclic groups can arise as a quotient $G/Z(G)$? Justify your answer.

3. Let X be a subset of a group G. The *centralizer* of X, denoted $C_G(X)$, is the set of all elements that commute with everything in X:

$$C_G(X) = \big\{ g \in G \mid gx = xg,\ \forall x \in X \big\}.$$

   If $X = \{x\}$, then we denote the centralizer as $C_G(x)$.
   (a) Show that $C_G(X)$ is a subgroup of G.
   (b) If H is a subgroup of G, show that $C_G(H) \trianglelefteq N_G(H)$.
   (c) Fix $x \in G$, and define the map

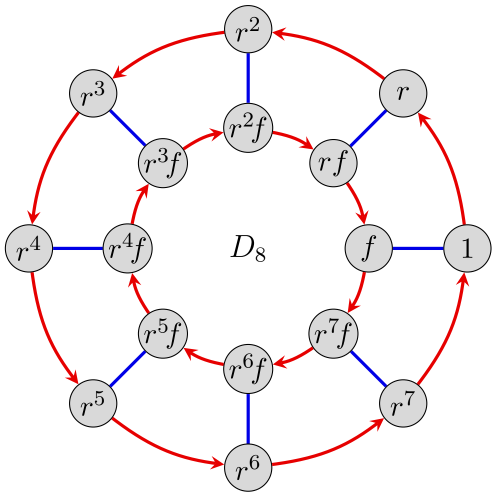
$$\phi\colon \big\{\text{left cosets of } C_G(x)\big\} \longrightarrow \big\{\text{conjugates of } x\big\}, \qquad \phi\colon gC_G(x) \longmapsto gxg^{-1}.$$

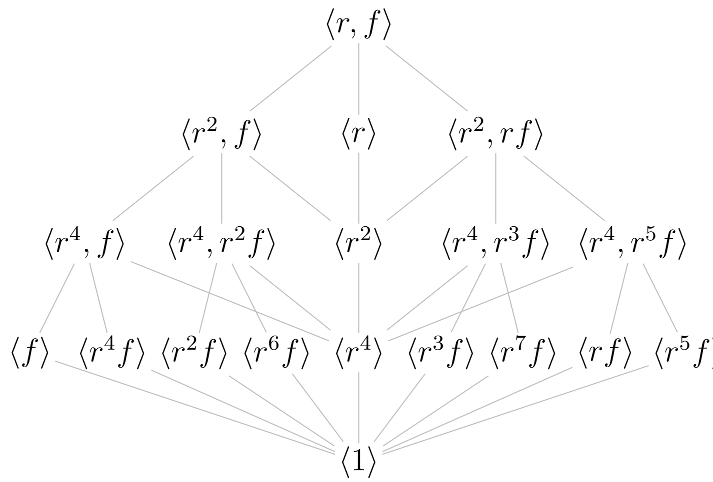
   Show that this map is well-defined and a bijection.
   (d) Derive the useful formula $|G| = |\operatorname{cl}_G(x)| \cdot |C_G(x)|$, for any $x \in G$.
   (e) For Q_8 and D_6, compute the centralizers of each element $x \in G$, as well as $N_G(\langle x \rangle)$. The partition of these groups by conjugacy classes is shown below.

1	i	j	k
-1	$-i$	$-j$	$-k$

1	r	r^2	f	r^2f	r^4f
r^3	r^5	r^4	rf	r^3f	r^5f

(f) Partition the elements of the group D_8 by conjugacy classes, and arrange them in a table, as above. Then repeat the previous part for this group. The Cayley graph and subgroup lattice for D_8 is shown below, for convenience.





4. Recall that two elements in S_n are conjugate if and only if they have the same cycle type.

(a) Determine how many elements there are of each cycle type in S_4, and in S_5. Note that the sum of your answers should add up to $|S_4| = 4! = 24$ and $|S_5| = 5! = 120$, respectively.

(b) Partition the elements of S_4 by conjugacy class.

(c) Compute the centralizers of e, (12), (123), (1234), and (12)(34) in S_4.

(d) Partition the elements of A_4 by conjugacy class. Then pick one element from each class, and find its centralizer. [*Hint:* two-thirds of the elements in A_4 are 3-cycles, so they cannot all be in the same conjugacy class.]

(e) Find the centralizer of each of the elements e, (12), (123), (1234), (12345), (12)(34), and (123)(45) in S_5.