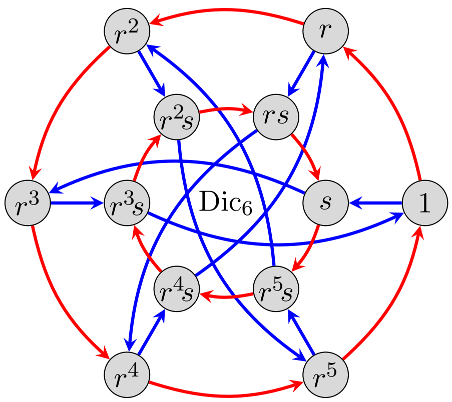
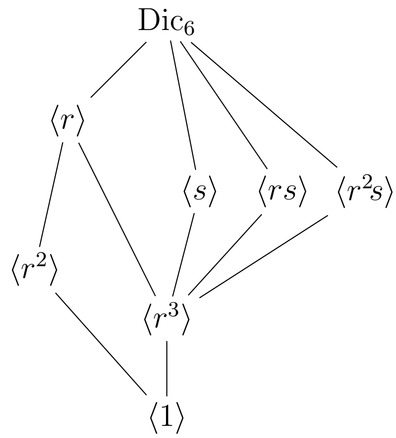
1. In this problem, we will explore the actions of the dicyclic group Dic_6 and its automorphism group on itself and its subgroups by conjugation. A Cayley diagram, subgroup lattice, and conjugacy classes are shown below.





rs	r^5s	r^3s
s	r^2s	r^4s
r^3	r^2	r^4
1	r	r^5

(a) The right action of Dic_6 on itself by conjugation is defined by the homomorphism

$$\phi\colon \text{Dic}_6 \longrightarrow \text{Perm}(S)\,, \qquad \phi(g) = \text{the permutation that sends each } x \mapsto g^{-1}xg.$$

Draw the action diagram and construct the fixed point table. Find $\text{stab}(s)$ for each $s \in S$, $\text{fix}(g)$ for each $g \in G$, as well as $\text{Ker}(\phi)$ and $\text{Fix}(\phi)$.

(b) The inner automorphism group

$$\text{Inn}(\text{Dic}_6) \cong \text{Dic}_6 / Z(\text{Dic}_6) = \text{Dic}_6 / \langle r^3 \rangle \cong D_3$$

also acts on Dic_6. Construct the action diagram and fixed point table of this action and find $\text{stab}(s)$, $\text{fix}(g)$, $\text{Ker}(\phi)$ and $\text{Fix}(\phi)$. Then draw the subgroup lattice of $\text{Inn}(\text{Dic}_6) = \langle \varphi_r, \varphi_s \rangle$, where $\varphi_g \colon x \mapsto g^{-1}xg$.

(c) The automorphism group of Dic_6 is $\text{Aut}(\text{Dic}_6) = \langle \varphi_r, \varphi_s, \omega \rangle$ acts on Dic_6, where ω is the outer automorphism defined by

$$\omega\colon \text{Dic}_6 \longrightarrow \text{Dic}_6, \qquad \omega(r) = r, \quad \omega(s) = s^{-1} = r^3 s,$$

that "reverses" the blue arrows. Make a diagram showing how each automorphism permutes the elements of Dic_6. Then construct the action diagram, fixed point table, and find $\text{stab}(s)$, $\text{fix}(g)$, $\text{Ker}(\phi)$ and $\text{Fix}(\phi)$.

(d) The automorphism group $\text{Aut}(\text{Dic}_6) = \langle \varphi_r, \varphi_s, \omega \rangle$ is isomorphic to D_6. Construct a Cayley diagram and subgroup lattice using these generators.

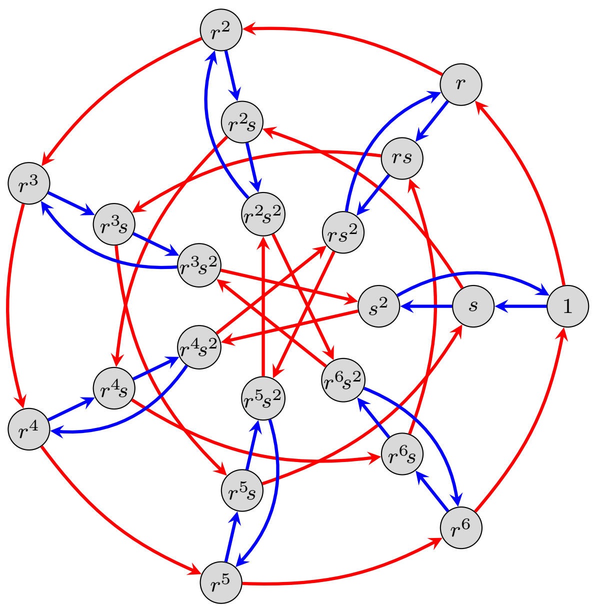
(e) The group $\text{Aut}(\text{Dic}_6)$ also acts on the conjugacy classes of Dic_6. Construct the action diagram, fixed point table, and find $\text{stab}(s)$, $\text{fix}(g)$, $\text{Ker}(\phi)$ and $\text{Fix}(\phi)$.

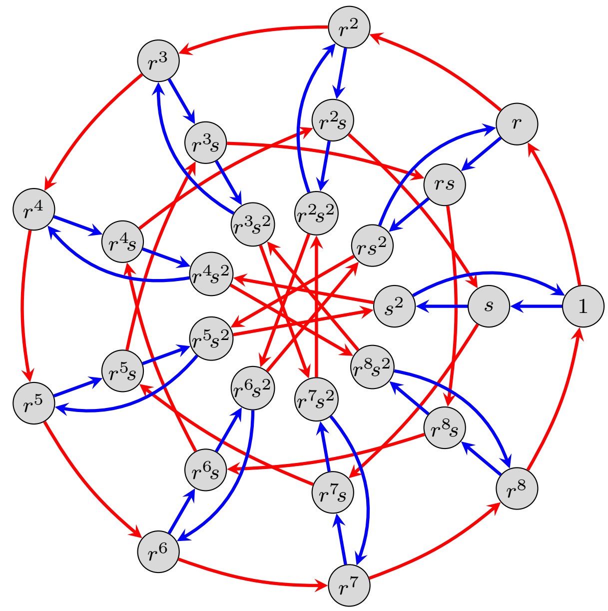
(f) The group Dic_6 acts on its subgroups by conjugation, via the homomorphism

$$\phi\colon \text{Dic}_6 \longrightarrow \text{Perm}(S)\,, \qquad \phi(g) = \text{the permutation that sends each } H \mapsto g^{-1}Hg.$$

Construct the action diagram superimposed on the subgroup lattice. Then construct the fixed point table and find $\text{stab}(s)$, $\text{fix}(g)$, $\text{Ker}(\phi)$ and $\text{Fix}(\phi)$.

2. Carry out the following steps for the groups $C_7 \rtimes C_3$ and $C_9 \rtimes C_3$, whose Cayley diagrams are shown below.





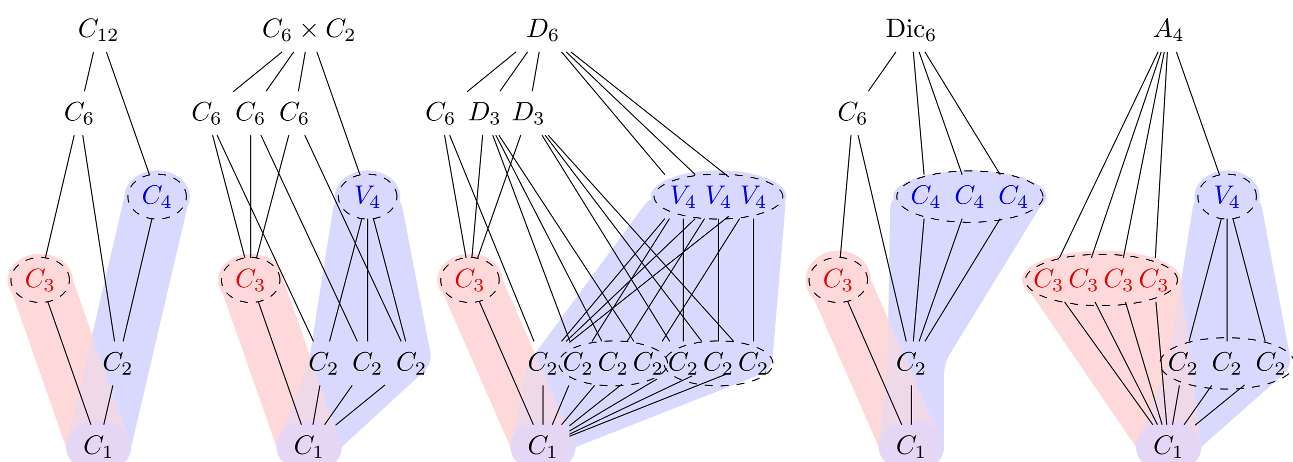
(a) Let G act on its subgroups by conjugation. Draw the action diagram superimposed on the subgroup lattice. Construct the fixed point table, and find $\text{stab}(H)$ for each $H \leq G$, $\text{Ker}(\phi)$ and $\text{Fix}(\phi)$.

(b) Let G act on the right cosets of $H = \langle s \rangle$, via the homomorphism

$$\phi \colon G \longrightarrow \text{Perm}(S)\,, \qquad \phi(g) = \text{the permutation that sends each } Hx \mapsto Hxg.$$

Construct the action diagram, fixed point table, and find $\text{stab}(Hx)$ for each right coset, $\text{Ker}(\phi)$ and $\text{Fix}(\phi)$.

3. Loosely speaking, the upcoming Sylow theorems will us that (1) all p-subgroups come in a single "p-subgroup tower", (2) the "top" of these towers are a single conjugacy class, and (3) the size of this class is 1 mod p. This is illustrated below with the groups of order 12.



Using the LMFDB, construct analogous diagrams for the groups of order 18 and 20.

4. Let A, B be subgroups of a group G.

   (a) Let $A\times B$ act on the set $AB \subseteq G$, via $(a,b)x = axb^{-1}$, which need not be a subgroup of G. Find the orbit and the stabilizer of the identity element $e = e\cdot e \in AB$. Use this, with the orbit-stabilizer theorem, to derive a formula for the size of the set $|AB|$.

   (b) For $x \in G$, define the (A,B)-*double coset* containing it to be the set

   $$AxB := \{axb \mid a \in A,\, b \in B\}.$$

   Show that G is the disjoint union of its (A,B)-double cosets. [*Hint*: One way is to show that $x \sim y$ iff $x \in AyB$ is an equivalence relation. Another way is to show that if $AxB \cap AyB \neq \emptyset$, then $AxB = AyB$.]

   (c) Partition the dihedral group D_3 into the (A,B)-double cosets of $A = \langle f\rangle$ and $B = \langle rf\rangle$. Does anything about this surprise you?

   (d) Let A act on the set G/B of left cosets of B by left multiplication. Show that $\mathrm{stab}(xB) = A \cap xBx^{-1}$, and then use the orbit-stabilizer theorem to determine the size of $\mathrm{orb}(xB)$.

   (e) Show that AxB is the union of exactly $[A : A \cap xBx^{-1}]$ left cosets of B in G.

   (f) Derive a formula for $|AxB|$.