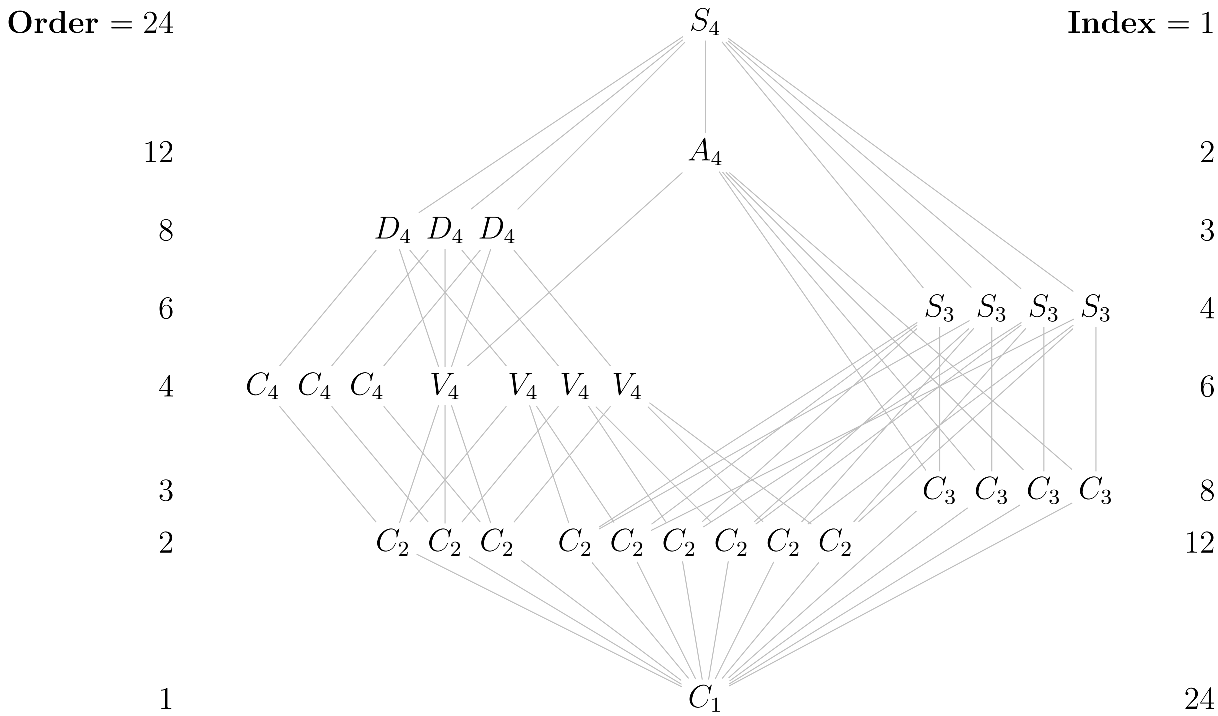
1. The subgroup lattice of the symmetric group $S_4$ is shown below.




(a) Partition the subgroups into conjugacy classes. Carefully and completely justify your answers using the Sylow theorems, without making reference to cycle type.

(b) For each conjugacy class $\mathrm{cl}_G(H)$, find the isomorphism type of the normalizer $N_G(H)$.

(c) Using the GroupNames website, make a table of all 15 groups of order 24, the number of subgroups, and basic information about their Sylow $p$-subgroups (number and isomorphism type). Write down at least one observation that you find interesting.

(d) Which groups are *not* an internal direct or semidirect product of Sylow subgroups?

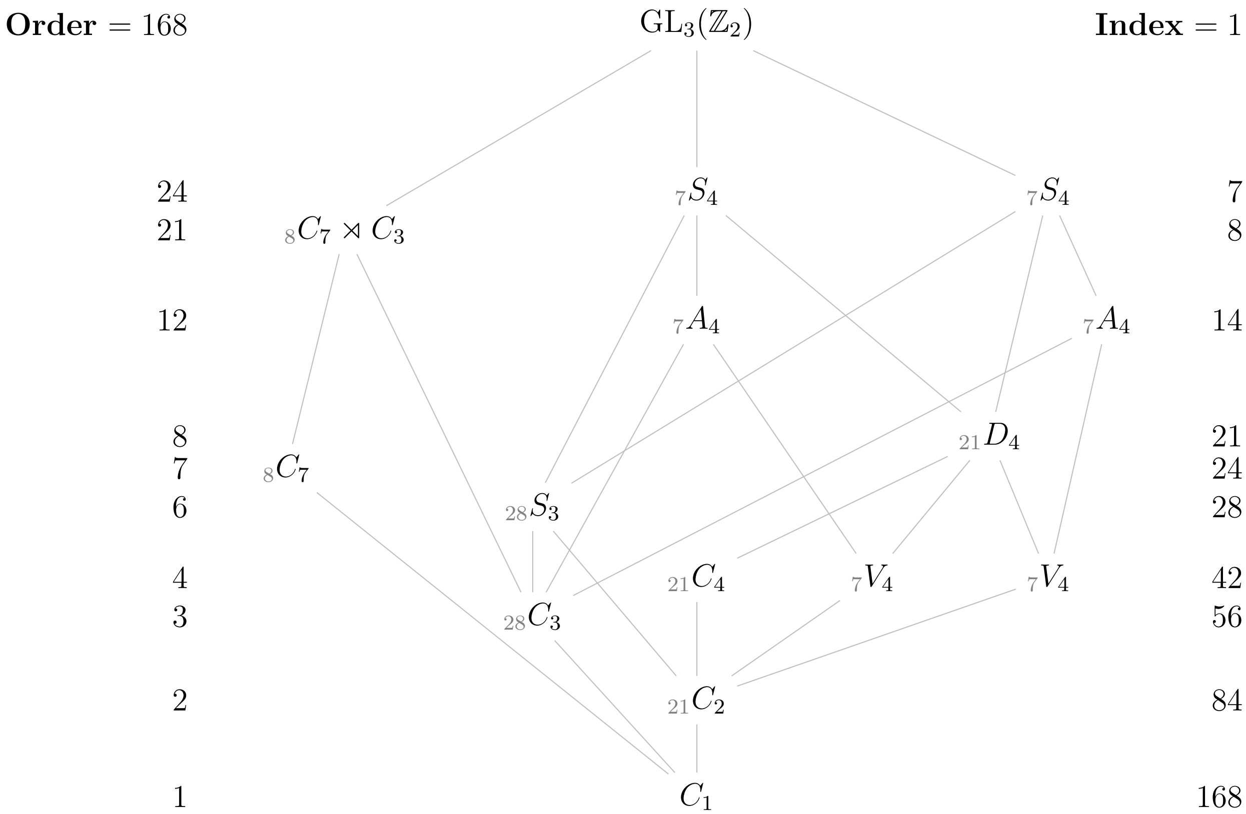
(e) None of the following groups are among the 15 listed on GroupNames: $D_6 \times C_2$, $C_6 \rtimes C_4$, $C_6 \rtimes C_2^2$, $C_4 \rtimes C_6$, $C_3 \rtimes C_2^3$, $C_2^3 \rtimes C_3$, $C_2^2 \rtimes C_6$, $C_3 \rtimes Q_8$, $Q_8 \rtimes C_3$, $C_4 \rtimes S_3$. Find which of the 15 each is isomorphic to, and add this this to your table under the "alias(es)" column.

2. Show that there are no simple groups of the following order.

(i) $45 = 3^2 \cdot 5$ (ii) $56 = 2^3 \cdot 7$ (iii) $108 = 2^2 \cdot 3^3$ (iv) $p^n$ $(n > 1)$.

[*Hint*: For Part (d), first use a suitable group action to show that $|Z(G)| > 1$.]

3. After $A_5$, the next smallest nonabelian simple group is $G = \mathrm{GL}_3(\mathbb{Z}_2)$, the invertible $3 \times 3$ binary matrices. It has order $168 = 2^3 \cdot 3 \cdot 7$, and its conjugacy poset is shown below.




(a) Color-code the $p$-subgroups, then draw arrows from each $\mathrm{cl}(H)$ to $\mathrm{cl}(N(H))$.

(b) Show that there is a non-trivial homomorphism $\phi\colon \mathrm{GL}_3(\mathbb{Z}_2) \to S_8$.

(c) Show that this homomorphism must be an embedding, and conclude that the order-40320 group $S_8$ has at least one subgroup isomorphic to $\mathrm{GL}_3(\mathbb{Z}_2)$.

(d) Show that every such subgroup of $S_8$ additionally must be contained in $A_8$.

4. In this problem, you will classify all groups of order 12, and then of order 18.

(a) Find all abelian groups of order 12, up to isomorphism.

(b) Let $G$ be a nonabelian group of order 12. Show that $G$ must have a normal Sylow $p$-subgroup, for some $p$.

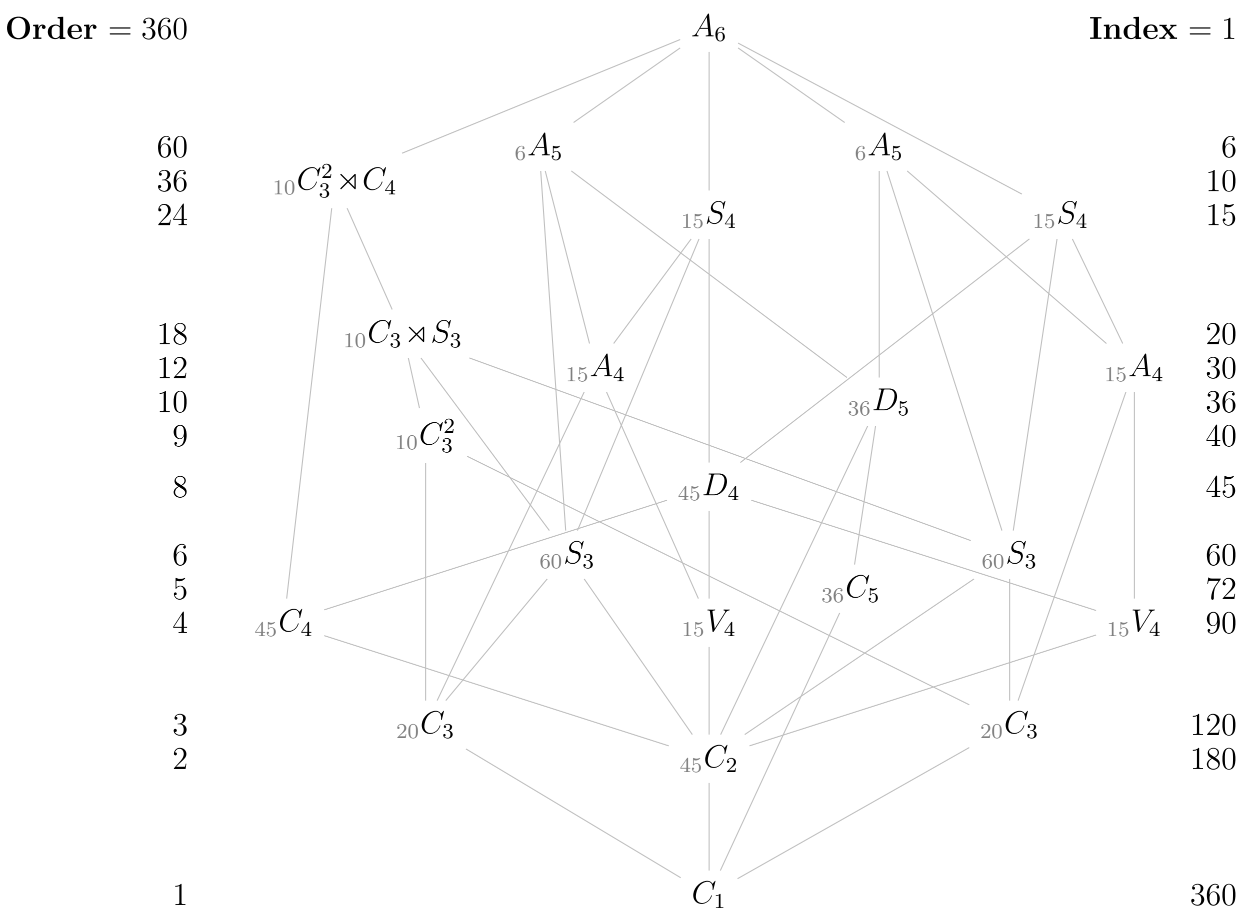
(c) Prove that $G$ must be a semidirect product of two of its Sylow $p$-subgroups.

(d) Find all semidirect products $G \cong P \rtimes_\theta Q$ by constructing all homomorphisms $\theta\colon Q \to \mathrm{Aut}(P)$, where $P$ and $Q$ are Sylow subgroups for different primes.

(e) There are three nonabelian groups of order 12: the alternating group $A_4$, the dihedral group $D_6$, and the dicyclic group $\mathrm{Dic}_6$. For each semidirect product in the previous part, determine which of these it is isomorphic to, with justification.

(f) Repeat Parts (a)–(d), and modify if necessary, to classify all groups of order 18.

5. The alternating group $A_6$ is the third smallest nonabelian simple group. It has order $6!/2 = 360 = 2^3 \cdot 3^2 \cdot 5$, and 501 subgroups contained in 22 conjugacy classes.




(a) Distinguish the $p$-subgroups by colors on the lattice.

(b) For each non-singleton conjugacy class $\mathrm{cl}(H)$, draw an arrow from it to $\mathrm{cl}(N(H))$, the conjugacy class of its normalizer.

(c) Now, let $G$ be an unknown group of order $90 = 2 \cdot 3^2 \cdot 5$.

(i) Show that if $G$ has a non-normal Sylow 5-subgroup, then there is a non-trivial homomorphism $\phi\colon G \to S_6$.

(ii) Show that if $\phi(G)$ is contained in the simple group $A_6$, then $\phi$ cannot be injective.

(iii) Explain why this implies that $G$ cannot be simple.

(iv) Find all possibilities for $n_2$, $n_3$, and $n_5$, where $n_p$ is the number of Sylow $p$-subgroups of $G$. Then, using GroupNames or LMFDB, make a list of all groups of order 90, and write down the actual vaules of $n_2$, $n_3$, and $n_5$ for each, as well as the isomorphism type of the Sylow 3-subgroup(s) – either $C_9$ or $C_3^2$. Does anything surprise you about this?