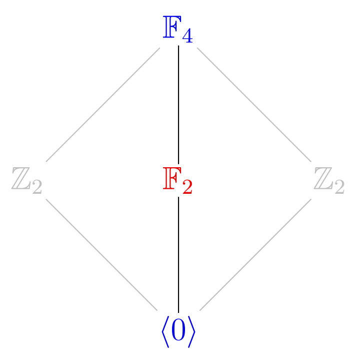
1. Construct the Cayley tables, Cayley graphs, and subring lattices of the finite field $\mathbb{F}_9 \cong \mathbb{Z}_3[x]/(x^2+x+2)$. Examples for the finite fields

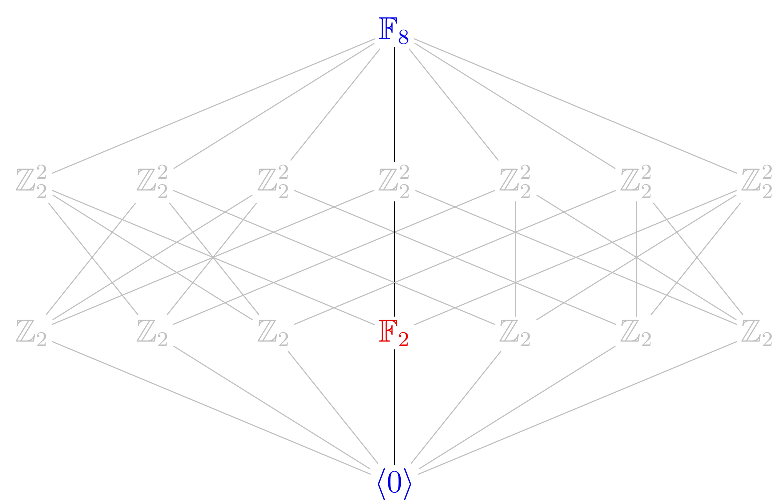
$$\mathbb{F}_4 \cong \mathbb{Z}_2[x]/(x^2+x+1) \qquad \text{and} \qquad \mathbb{F}_8 \cong \mathbb{Z}_2[x]/(x^3+x+1)$$

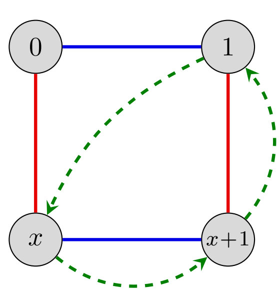
are shown below.

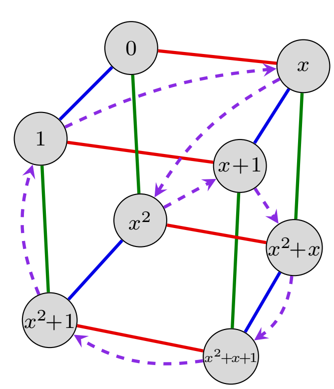
$\times$	1	x	$x+1$
1	1	x	$x+1$
x	x	$x+1$	1
$x+1$	$x+1$	1	x

$\times$	1	x	$x+1$	x^2	x^2+1	x^2+x	x^2+x+1
1	1	x	$x+1$	x^2	x^2+1	x^2+x	x^2+x+1
x	x	x^2	x^2+x	$x+1$	1	x^2+x+1	x^2+1
$x+1$	$x+1$	x^2+x	x^2+1	x^2+x+1	x^2	1	x
x^2	x^2	$x+1$	x^2+x+1	x^2+x	x	x^2+1	1
x^2+1	x^2+1	1	x^2	x	x^2+x+1	$x+1$	x^2+x
x^2+x	x^2+x	x^2+x+1	1	x^2+1	$x+1$	x	x^2
x^2+x+1	x^2+x+1	x^2+1	x	1	x^2+x	x^2	$x+1$









2. All of the isomorphism theorems for groups have analogues for rings. The proofs just amount to showing that the group homomorphisms are additionally ring homomorphisms, and we will carry out these details in this problem.

   (a) The *fundamental homomorphism theorem* (FHT) says that $R/\operatorname{Ker}(\phi) \cong \operatorname{Im}(\phi)$. To prove this, we construct a map

   $$\iota\colon R/I \longrightarrow \operatorname{Im}(\phi), \qquad \iota(r+I) = \phi(r),$$

   which we already know is a well-defined group isomorphism. Show that it is also a ring homomorphism.

   (b) By the *correspondence theorem*, every subgroup of R/I has the form J/I for some $I \leq J \leq R$. Show that J/I is an ideal of R/I if and only if J is an ideal of R.

   (c) The *fraction theorem* says that $(R/I)/(J/I) \cong R/J$. This is proven by constructing a map

   $$\phi\colon R/I \longrightarrow R/J, \qquad \phi(r+I) = r+J$$

   which we already know is a group homomorphism with $\operatorname{Ker}(\phi) = J/I$. Show that it is also a ring homomorphism, and then apply the FHT.

   (d) The *diamond theorem* says that $(S+I)/I \cong S/(S \cap I)$ for a subring S and ideal I.

      (i) Prove that $S \cap I$ is an ideal of S.

      (ii) We already know that

      $$\phi\colon S \longrightarrow (S+I)/I, \qquad \phi(s) = s+I$$

      is a group homomorphism with $\operatorname{Ker}(\phi) = S \cap I$. Show that it is also a ring homomorphism, and then apply the FHT.

3. Use Zorn's lemma to show that the ring $\mathbb{R}$ contains a subring A containing 1 that is maximal with respect to the property that $1/2 \notin A$.

4. Let R be a commutative ring with 1.

   (a) Show that if x is contained in every maximal ideal, then $1+x$ is a unit.

   (b) A ring is *local* if it has a unique maximal ideal. Show that R is local if and only if the non-units form an ideal.

   (c) Let p be a fixed prime. Show that the ring

   $$R = \left\{ \tfrac{a}{b} \mid a, b \in \mathbb{Z},\ (a,b) = 1,\ p \nmid b \right\} \subseteq \mathbb{Q}$$

   is local, and characterize units and maximal unique ideal.