Math 4120, Spring 2024

Study guide: Midterm 1.

Note: This is just a guide, not an all-inclusive list.

Definitions to know.

(1) A *group* G. (The "official" definition.)
(2) The *order* of an element $g \in G$.
(3) A *left coset* xH of a subgroup $H \leq G$.
(4) A *normal subgroup* $H \trianglelefteq G$.
(5) The *index* $[G : H]$ of a subgroup $H \leq G$.
(6) The *direct product* $A \times B$ of two groups A and B.
(7) The *quotient* G/H of a group G by a normal subgroup $H \trianglelefteq G$.
(8) The *normalizer* $N_G(H)$ of a subgroup $H \leq G$.
(9) The *center* $Z(G)$ of a group.
(10) What it means for multiplication $aH \cdot bH := abH$ in the quotient group G/H to be *well-defined*.
(11) The *conjugacy class* $\mathrm{cl}_G(x)$ of an element $x \in G$, and the conjugacy class $\mathrm{cl}_G(H)$ of a subgroup.
(12) The *centralizer* $C_G(x)$ of an element $x \in G$.

Cayley diagrams and presentations.

(1) Be able to use a Cayley diagram as a "group calculator", e.g., multiply elements and find their inverses.
(2) Be able to construct Cayley diagrams of V_4, C_n, D_n, Q_8, Dic_n, SD_8, SA_8, and write a group presentation for these groups.
(3) Given an unknown Cayley diagrams, write a group presentation that describes it.
(4) Be able to identify left and right cosets from a Cayley diagram.
(5) Be able to find the normalizer of a subgroup from a Cayley diagram.

Cycle diagrams.

(1) Be able to construct a cycle diagram of groups like C_n, D_n, and Q_8.
(2) Use a cycle diagram to identity all cyclic subgroups, the first step in constructing a subgroup lattice.

Subgroup lattices.

(1) Be able to construct the subgroup lattices of $\mathbb{Z}_n$, V_4, $D_3 \cong S_3$, D_4, D_5, Q_8, $\mathbb{Z}_4 \times \mathbb{Z}_2$, and A_4.
(2) Be able to label the edges of a subgroup lattice with the index, $[H : K]$.
(3) Know how to be "fluent" reading subgroup lattices. For example, given H and K, where to find $H \cap K$ and $\langle H \cup K \rangle$, and how to identify when a subgroup is normal (e.g., G, $\{e\}$, index-2 subgroups, and unicorns).
(4) Be able to determine the normalizer of H on a Cayley diagram, given knowledge of its conjugacy class, or vice-versa.

Helpful misc. facts about familiar groups.

(1) The cyclic group C_n is generated by r^k, iff $\gcd(n, k) = 1$.
(2) $C_n \times C_m \cong C_{nm}$ iff $\gcd(n, m) = 1$.
(3) Every subgroup of Q_8 is normal.
(4) The dihedral group D_n has n or $n + 1$ elements of order 2, depending on the parity of n. It can be generated by a rotation and reflection, or two adjacent reflections.
(5) The dihedral group D_n is a semidirect product $C_n \rtimes_\theta C_2$.

(6) There is one frieze group that needs three symmetries to generate it. It contains three non-abelian frieze groups (the "infinite dihedral group") as subgroups: (i) removing all horizontal reflections, (ii) remove all 180°-rotations, or (iii) remove half of each of these.
(7) Know how to represent the groups V_4, C_n, D_n, Q_n, and Dic_n with 2×2 matrices.
(8) Two canonical generatating sets for the symmetric group: $S_n = \langle (12), (123\cdots n)\rangle = \langle (12), (23), \dots, (n{-}1\ n)\rangle$.
(9) Know the difference between *minimal* and *minimum* generating sets.
(10) The automorphism group $\mathrm{Aut}(C_n)$ (of "rewirings") is isomorphic to the group

$$U_n = \big\{k \mid 1 \leq k < n,\ \gcd(n,k) = 1\big\}.$$

(11) Know how to construct the Cayley diagram of $\mathrm{Aut}(C_n)$, and a semidirect product, given a "labeling map" $\theta\colon H \to \mathrm{Aut}(C_n)$.

Useful facts and techniques.

(1) Two different ways to show that a subset $H \subseteq G$ is a subgroup.
(2) Three different ways to show that a subgroup $H \leq G$ is normal.
(3) Know to how compose permutations in cycle notation, and find inverses, e.g., $(123\cdots n)^{-1} = (1n\cdots 32)$.
(4) Know which permutations are even vs. odd.
(5) Learn to classify all finite abelian groups of a fixed order.
(6) Two elements in S_n are conjugate iff they have the same cycle type.
(7) If n is odd, then all reflections in D_n are conjugate. If n is even, then there are two conjugacy classes of reflections.
(8) $\mathrm{cl}_G(x) = \{x\}$ if and only if $x \in Z(G)$.
(9) $\mathrm{cl}_G(H) = \{H\}$ if and only if $H \trianglelefteq G$.
(10) Use the fact that $|\,\mathrm{cl}_G(x)| = [G : C_G(x)]$ to help partition G by conjugacy classes, and/or find the centralizer.
(11) Use the fact that $|\,\mathrm{cl}_G(H)| = [G : N_G(H)]$ to help partition G's subgroups by conjugacy classes, and/or find the normalizer.

Proofs to learn.

(1) Show that the identity element of a group is unique.
(2) Show that every element in a group has a unique inverse.
(3) Show that if $\{H_\alpha \mid \alpha \in A\}$ is a collection of subgroups, then $\displaystyle\bigcap_{\alpha \in A} H_\alpha$ is a subgroup.
(4) Show that $xH = H$ if and only if $x \in H$.
(5) Show that if $[G : H] = 2$, then $H \trianglelefteq G$.
(6) Prove that if $K \leq H \leq G$ and $K \trianglelefteq G$, then $K \trianglelefteq H$.
(7) Show that the center $Z(G) = \{z \in G \mid gz = zg,\ \forall g \in G\}$ is a subgroup of G and that it is normal.
(8) Let $H \trianglelefteq G$. Prove that multiplication of cosets is well-defined: if $a_1H = a_2H$ and $b_1H = b_2H$, then $a_1H \cdot b_1H = a_2H \cdot b_2H$. Additionally, show that G/H is a group under this binary operation.
(9) The tower law: $[G : H][H : K] = [G : K]$.
(10) Prove that if G is abelian and $H \leq G$, then G/H is abelian.
(11) Show that the normalizer $N_G(H) = \{g \in G \mid gHg^{-1} = H\}$ is a subgroup of G.
(12) Show that if $A, B \leq G$, and A normalizes B, then AB is a subgroup of G.

Study guide: Midterm 2.

Definitions to know.

1. A *homomorphism* ϕ from a group G to a group H.
2. What it means for a homomorphism to be an *embedding* and a *quotient*.
3. An *isomorphism* $\phi\colon G \to H$.
4. An *automorphism* $\phi\colon G \to H$.
5. The *kernel* of a homomorphism $\phi\colon G \to H$.
6. What it means for a map $f\colon G/N \to H$ to be *well-defined*.
7. The *commutator subgroup* G' of a group G, and the *abelianization* G/G'.
8. An *inner automorphism* and *outer automorphism* of G.
9. A *group action* of G on a set S.
10. Local features of an action: the *orbit* $\mathrm{orb}(s)$ and *stabilizer* $\mathrm{stab}(s)$ of $s \in S$, and the *fixator* $\mathrm{fix}(g)$ of $g \in G$.
11. Global features of an action: the set $\mathrm{Fix}(\phi)$ of *fixed points*, and the *kernel* $\mathrm{Ker}(\phi)$.
12. A p*-group*, and a *Sylow* p*-subgroup* of a group G.

Useful facts and techniques.

1. Be able to show that a certain map is a homomorphism, using the definition.
2. A homomorphism is 1-to-1 iff $\mathrm{Ker}(\phi) = \langle 1 \rangle$.
3. There are two ways to prove that $G/N \cong H$: Either construct a map $G/N \to H$ and prove it is a well-defined bijective homorphism, or construct a map $\phi\colon G \to H$ and prove it is an onto homomorphism with $\mathrm{Ker}(\phi) = N$.
4. Learn the statement of the correspondence theorem: there is a 1–1 correspondence between subgroup of G/N and subgroups of G that contain N. Moreover, every subgroup of G/N is of the form H/N for some $N \leq H \leq G$. Be able to interpret this visually in terms of subgroup lattices.
5. Be able to recongize subgroups and quotients of a group simply from the subgroup lattice: subgroups appears as "stagmites", and quotients as "stalactites."
6. Learn how to identify the commutator subgroup of G and abelinization G/G' just from the subgroup lattice.
7. The automorphism group of a cyclic group is $\mathrm{Aut}(\mathbb{Z}_n) \cong U_n$, the multiplitive group of integers modulo n.
8. Inner automorphism have the form $\varphi_g\colon x \mapsto gxg^{-1}$. The inner automorphism group of G is $\mathrm{Inn}(G) \cong G/Z(G)$. That is, $\varphi_g = \varphi_h$ iff g and h are in the same cosets of $Z(G)$.
9. Given only a subgroup lattice of G, be able to determine whether G is isomorphic to the semidirect product, or direct product, of two of its subgroups.
10. The orbit-stabilizer theorem: If G acts on S, then $|G| = |\mathrm{orb}(s)| \cdot |\mathrm{stab}(s)|$ for any $s \in S$.
11. The orbit counting theorem: the average size of $\mathrm{fix}(g)$ is the number of orbits.
12. Learn the local featuers (orbits, stabilizers, fixed point sets), and global features (kernel, set of fixed points) for each of the following actions: following actions:
    - (i) G acting on itself by right multiplication.
    - (ii) G acting on itself by conjugation.
    - (iii) G acting on its subgroups by conjugation.
    - (iv) G acting on its right cosets by right multiplication.
13. Constructing the "fixed point table" of an action, and identifying the features of an action from it.
14. Learn how to use the 3rd Sylow theorem to show that a group of a certain order is simple. (Usually, by showing that $n_p = 1$ for some prime p.)

Proofs to learn.

1. If $\phi\colon G \to H$ is a homomorphism, then $\phi(1_G) = 1_H$.
2. If $\phi\colon G \to H$ is a homomorphism, then $\phi(g^{-1}) = \phi(g)^{-1}$ for all $g \in G$.

(3) If G is abelian, then so is G/H.
(4) If $G/Z(G)$ is cyclic, then G is abelian (and hence $G/Z(G)$ is the trivial group).
(5) The kernel of any homomorphism is a subgroup, and is normal.
(6) Given a homomorphism $\phi\colon G \to H$, each preimage $\phi^{-1}(h)$ is a coset of $\mathrm{Ker}(\phi)$.
(7) $A \times B \cong B \times A$.
(8) If $H \leq G$, then $xHx^{-1} \cong H$ for any $x \in G$.
(9) There is no embedding $\varphi\colon \mathbb{Z}_n \to \mathbb{Z}$.
(10) If $\varphi\colon G \to H$ is a homomorphism and $N \trianglelefteq H$, then $\varphi^{-1}(N)$ is a normal subgroup of G.
(11) If $H \leq G$ is the only subgroup of G of order $|H|$, then it must be normal.
(12) The FHT: if $\phi\colon G \to H$ is a homomorphism, then $G/\mathrm{Ker}(\phi) \cong \mathrm{Im}(\phi)$.
(13) The correspondence theorem: every subgroup of G/N is has the form H/N, for some $H \leq G$ that contains N.
(14) The freshman theorem: given a chain $N \leq H \leq G$ of normal subgroups of G, $(G/N)/(H/N) \cong G/H$.
(15) The diamond isomorphism theorem: if A normalizes G, then $AB \leq G$, $B \trianglelefteq AB$, $(A \cap B) \trianglelefteq A$, and $AB/B \cong A/(A \cap B)$.
(16) Use the FHT to show that $|NH| = |N| \cdot |H|/|N \cap H|$.
(17) Show that $\mathbb{Q}^* \cong \mathbb{Q}^+ \times C_2$ and $\mathbb{Q}^*/\langle -1 \rangle \cong \mathbb{Q}^+$, where $\mathbb{Q}^*$ is the nonzero rationals under multiplication, and $\mathbb{Q}^+ \leq \mathbb{Q}^*$ is the subgroup of positive rationals.
(18) Show that G is abelian iff its commutator subroup $G' = \{e\}$.
(19) Show that G/G' is abelian.
(20) Show that $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.
(21) Use the FHT to show that $G/Z(G) \cong \mathrm{Inn}(G)$.
(22) Show that if G acts on S, then $\mathrm{stab}(s)$ is a subgroup of G, for any $s \in S$.
(23) Show that if G is a p-group, then $|Z(G)| > 1$.
(24) Show how Cayley's theorem follows from the orbit-stabilizer theorem, and a group acting on itself by multiplication.
(25) Show that if G has no subgroup of index 2, then any subgroup of index 3 is normal.
(26) Show that of $[G : H] = p$ for the smallest prime dividing $|G|$, then $H \trianglelefteq G$.

Study guide: Final exam.

Note: This is *in addition*, not instead, of the Midterm 1 and 2 material.

Definitions to memorize.

(1) A *ring* R.
(2) A *unit*, and a *zero divisor* of a ring.
(3) An *ideal* of a ring R (left, right, and two-sided).
(4) Types of rings: integral domain, division ring, field.
(5) The *quotient ring* R/I for some two-sided ideal I, and how to add and multiply elements.
(6) A *homomorphism* ϕ from a ring R to a ring S.
(7) A *maximal ideal* and a *prime ideal* of a ring R.

Useful facts and techniques.

(1) Construct the subring lattice of a small finite ring, and be able to determine the ideals, subrings that aren't ideals, and subgroups that aren't subrings.
(2) Know examples of ideals, subrings that aren't ideals, and subgroups that aren't subrings, in various rings.
(3) Know examples of both maximal ideals and prime ideals, prime ideals that aren't maximal.
(4) Learn how to construct a finite field $\mathbb{F}_q$ of order $q = p^k$.
(5) Know the statements of the fundamental homomorphism theorem and the correspondence theorem for rings and how to apply them.

Proofs to learn.

(1) If an ideal I of R contains a unit, then $I = R$.
(2) The FHT for rings: if $\phi\colon R \to S$ is a ring homomorphism, then $\mathrm{Ker}(\phi)$ is an ideal of R and $R/\mathrm{Ker}(\phi) \cong \mathrm{Im}(\phi)$.
(3) Prove the isomorphism theorems for rings, assuming the results for groups.
(4) The following are equivalent for commutative rings: (i) I is a maximal ideal, (ii) R/I is simple, (iii) R/I is a field.
(5) An ideal P is prime iff R/P is an integral domain.
(6) A ring R is an integral domain iff 0 is a prime ideal.
(7) Every maximal ideal is prime.
(8) Use Zorn's lemma to show that every ideal is contained in a maximal ideal.