

Chapter 7: Universal constructions

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Math 4130, Visual Algebra

Factoring maps

We've discussed a number of properties that can be described as

“the minimal _____,” or “the maximal _____,”

satisfying some condition.

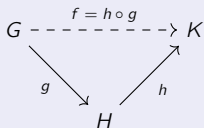
We'll see to express this concisely in terms of maps and commutative diagrams.

This will highlight similarities and patterns that are inherent in seemingly different structures, streamline proofs, and lead to new insight.

Warm-up exercise (easy)

Given maps $g: G \rightarrow H$ and $h: H \rightarrow K$, their composition $f := h \circ g$ is a map from G to K .

I.e., there is *always* a map $f: G \rightarrow K$ making the following diagram commute:



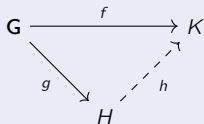
Factoring maps

Let's now consider two variants of the previous commutative diagram.

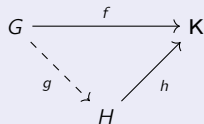
Definition

Given two maps...

1. *from the same domain*, $f: G \rightarrow K$, $g: G \rightarrow H$, when does there exist $h: H \rightarrow K$
 2. *into the same codomain*, $f: G \rightarrow K$, $h: H \rightarrow K$, when does there exist $g: G \rightarrow H$
- such that $f = h \circ g$?



" f factors through g "



" f factors through h "

We say that h and g are **factors** of f .

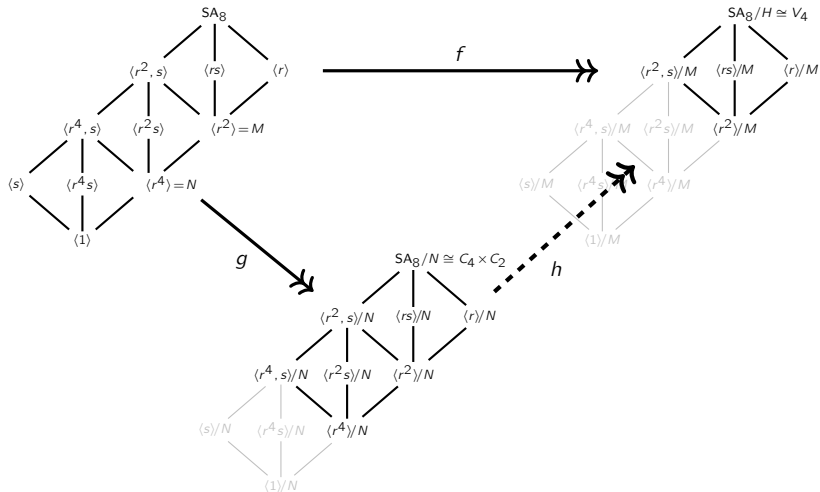
We'll do an example of each that will nicely illustrate when and why this happens.

Both will involve $G = \text{SA}_8 = \langle r, s \rangle$, and its subgroups $N = \langle r^2 \rangle \cong C_4$, and $M = \langle r^4 \rangle \cong C_2$.

Factoring maps: a quotient between the codomains

Let $G = SA_8 = \langle r, s \rangle$, and $N = \langle r^2 \rangle \cong C_4$, and $M = \langle r^4 \rangle \cong C_2$.

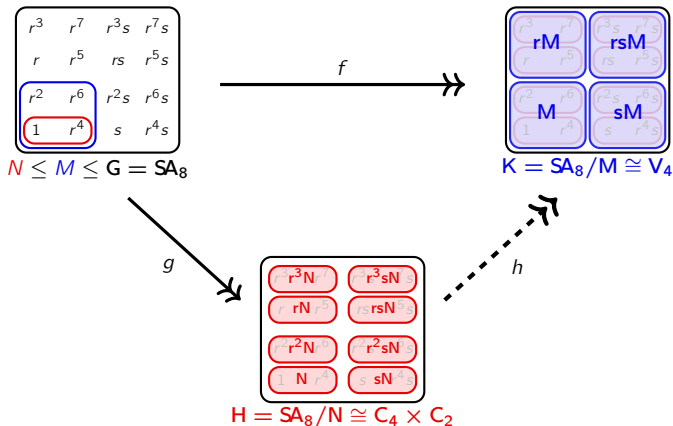
The standard quotient map $f: SA_8 \rightarrow V_4$ can be factored:



Factoring maps: a quotient between the codomains

Formally, this map is defined by

$$h: SA_8/N \longrightarrow SA_8/M, \quad h: gN \longmapsto gM.$$

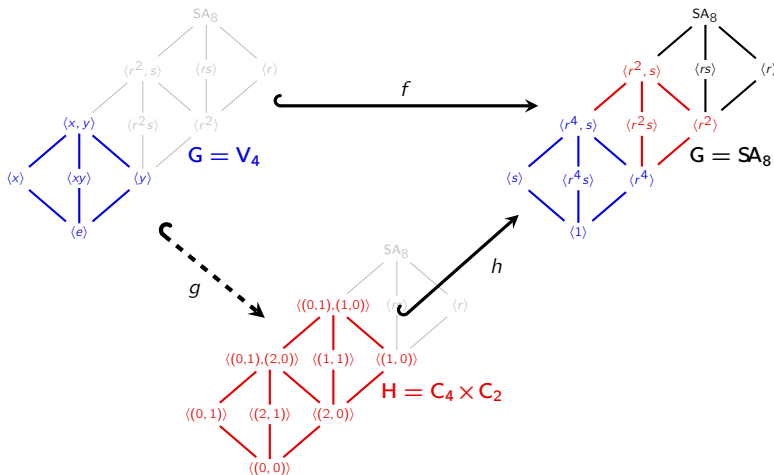


Factoring maps: an embedding between the domains

Let $V_4 = \{e, x, y, xy\}$ and $SA_8 = \langle r, s \rangle$. The embedding

$$f: V_4 \hookrightarrow SA_8, \quad x \mapsto s, \quad y \mapsto r^4$$

uniquely factors through $h: C_4 \times C_2 \rightarrow SA_8$, where $(1, 0) \mapsto r^2$ and $(0, 1) \mapsto s$.

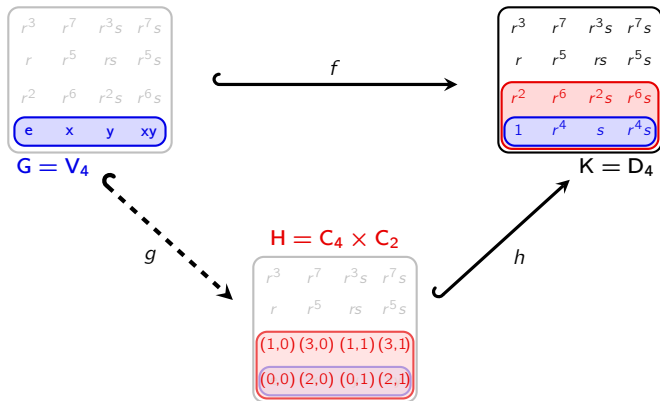


Factoring maps: an embedding between the domains

Let $V_4 = \{e, x, y, xy\}$ and $SA_8 = \langle r, s \rangle$. Here's that same embedding

$$f: V_4 \hookrightarrow SA_8, \quad x \mapsto r^4, \quad y \mapsto s$$

that uniquely factors through $h: C_4 \times C_2 \rightarrow SA_8$, where $(1, 0) \mapsto r^2$ and $(0, 1) \mapsto s$.

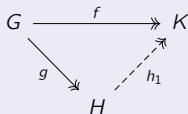


Canceling maps: when does existence imply uniqueness?

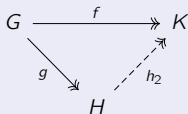
Proposition (“cancelation laws”)

Suppose we have functions $g_i: G \rightarrow H$ and $h_i: H \rightarrow K$ between sets, for $i = 1, 2$.

If g is surjective, then it right-cancels: $h_1 \circ g = h_2 \circ g \implies h_1 = h_2$.



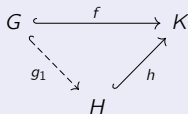
and



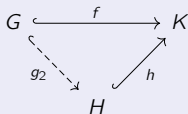
\implies

$$h_1 = h_2.$$

If h is injective, then it left-cancels: $h \circ g_1 = h \circ g_2 \implies g_1 = g_2$.



and



\implies

$$g_1 = g_2.$$

Key idea

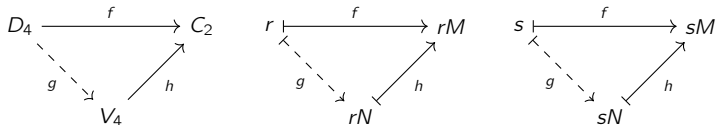
Injective functions have left inverses; surjective functions have right inverses.

Failure of uniqueness: a quotient between domains

Let $D_4 = \langle r, s \rangle$ with subgroups $N = \langle r^2 \rangle \cong C_2$ and $M = \langle r \rangle \cong C_4$.

Their quotients are $V_4 \cong D_4/N = \{N, rN, sN, rsN\}$ and $C_2 \cong D_4/M = \{M, sM\}$.

Define the functions $f: D_4 \rightarrow C_2$ and $h: V_4 \rightarrow C_2$ as follows:



We must have $g: 1 \mapsto N$. Since $f: r \mapsto M$, then $g(r) \in \text{Ker}(h) = \{rN, N\}$.

If $g(r) = N$, then g is not surjective, but we still have $f = h \circ g$.

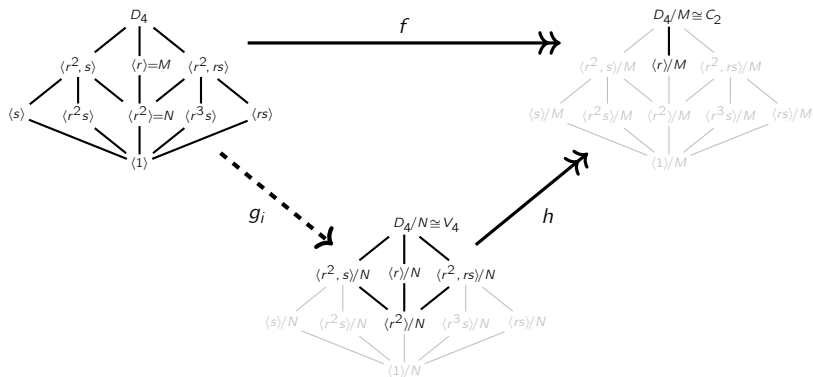
Warning!

The homomorphism $g: D_4 \rightarrow V_4$ is *not* uniquely defined! ($r \mapsto N$ would work too)

Moral: commutative diagrams can be deceiving!

Failure of uniqueness: a quotient between domains

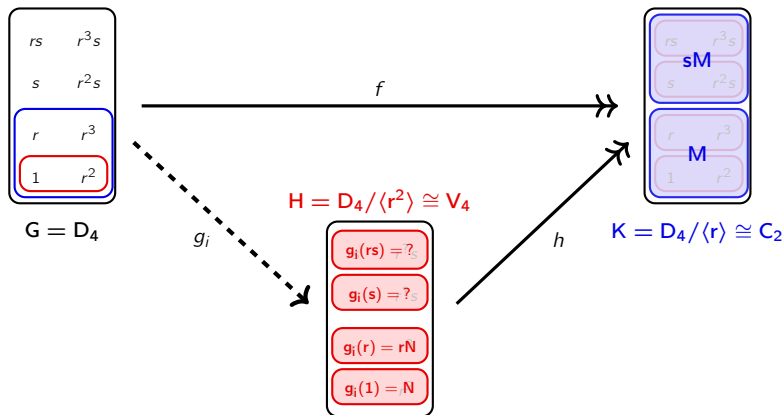
Multiple maps g_i make this diagram commute; both $r \mapsto rN$ and $r \mapsto N$ work.



For surjective maps, $h \circ g_1 = h \circ g_2 \not\Rightarrow g_1 = g_2$.

Failure of uniqueness: a quotient between domains

Note that $g_i: D_4 \rightarrow V_4$ need not be surjective for the following diagram to commute.



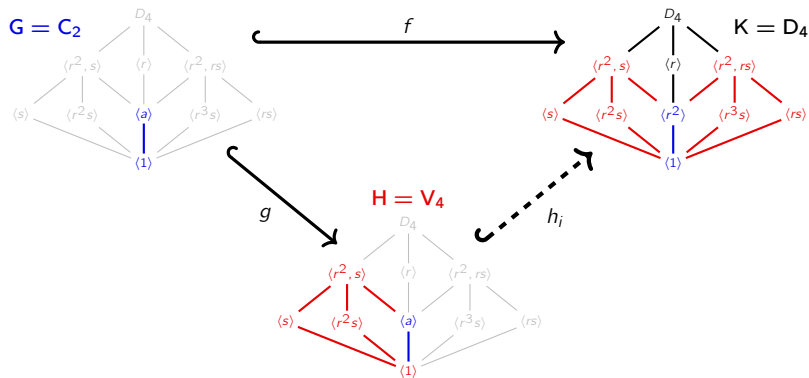
Any choice of $g_i(r) \in \{N, rN\}$ and $g_i(s) \in \{sN, rsN\}$ would work.

Failure of uniqueness: an embedding between codomains

Consider two maps from $G = C_2 = \{1, a\}$ into $H = V_4 = \{e, x, y, xy\}$ and $K = D_4 = \langle r, s \rangle$:

$$f: C_2 \longrightarrow D_4, \quad f(a) = r^2, \quad g: C_2 \longrightarrow V_4, \quad f(a) = x.$$

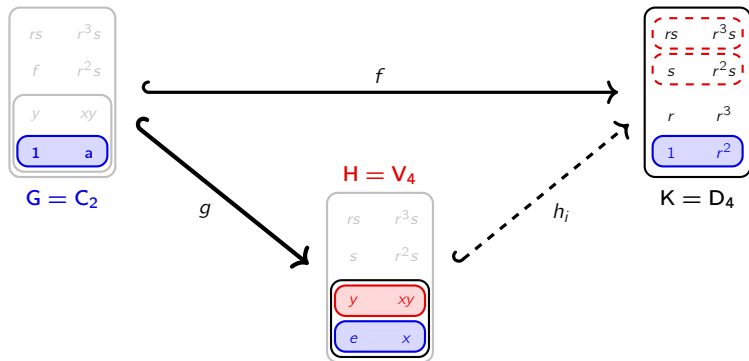
There are multiple embeddings $h_i: V_4 \hookrightarrow D_4$ that make this diagram commute:



For injective maps, $h_1 \circ g = h_2 \circ g \not\Rightarrow h_1 = h_2$.

Failure of uniqueness: an embedding between codomains

Here is another way to see why there are multiple embeddings $h_i: V_4 \hookrightarrow D_4$ that make this diagram commute:



For injective maps, $h_1 \circ g = h_2 \circ g \not\Rightarrow h_1 = h_2$.

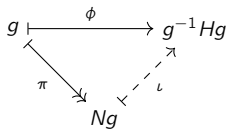
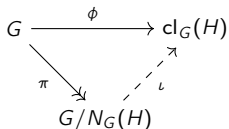
Factoring non-homomorphisms

Definition

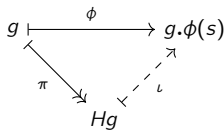
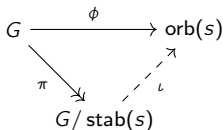
Let G/N be a set (not necessarily a group) of equivalence classes. The map ϕ from G **descends to a map from G/N** if it factors through the canonical quotient $\pi: G \rightarrow G/N$.

For example, we have seen that:

- the map $\phi: G \rightarrow \text{cl}_G(H)$ **descends to a bijection** $G/N_G(H) \rightarrow \text{cl}_G(H)$.



- the map $\phi: G \rightarrow \text{cl}_G(g)$ **descends to a bijection** $G/C_G(g) \rightarrow \text{cl}_G(g)$.
- For a fixed $s \in S$, $\phi: G \rightarrow \text{orb}(s)$ **descends to a bijection** $G/\text{stab}(s) \rightarrow \text{orb}(s)$.



Motivating the co-universal property of quotient groups

Definition

Given $H \leq G$, the **canonical inclusion map** is

$$\iota: H \hookrightarrow G, \quad \iota: h \mapsto h.$$

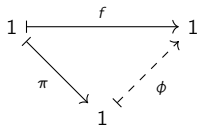
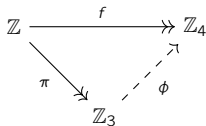
If $H \trianglelefteq G$, the **canonical quotient map** is

$$\pi: G \longrightarrow G/H, \quad \pi: g \mapsto gH.$$

There does not exist a homomorphism $\phi: \mathbb{Z}_3 \rightarrow \mathbb{Z}_4$ with $\phi(1) = 1$. To formalize this:

the canonical quotient $f: \mathbb{Z} \rightarrow \mathbb{Z}_4$ does not factor through $g: \mathbb{Z} \rightarrow \mathbb{Z}_3$.

That is, there does *not* exist $\phi: \mathbb{Z}_3 \rightarrow \mathbb{Z}_4$ making this diagram commute:

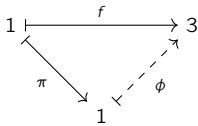
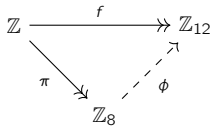


Preview: such a map exists iff $\text{Ker}(\pi) \leq \text{Ker}(f)$, i.e., f collapses at least as much as π .

Motivating the co-universal property of quotient groups

Does $\phi: \mathbb{Z}_8 \rightarrow \mathbb{Z}_{12}$, where $\phi(1) = 3$, define a homomorphism?

Is there a homomorphism ϕ making the following diagram commute?



Note that $\text{Ker}(f) = 4\mathbb{Z}$ is a subgroup of $\text{Ker}(\pi) = 8\mathbb{Z}$, and so f factors through π .

Not only does ϕ exist, it is automatically unique by the [cancellation laws](#).

The co-universal property of quotient groups

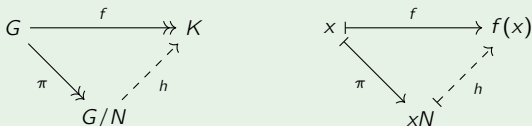
Theorem

Let $N \trianglelefteq G$ and $f: G \rightarrow K$ be a homomorphism such that $N \leq \text{Ker}(f)$. Then

1. f uniquely factors through $\pi: G \rightarrow G/N$ (i.e., $\exists! h: G/N \rightarrow K$ such that $g = h \circ \pi$).
2. h is injective iff $\text{Ker}(f) = N$.

Proof (i)

Assume WLOG that f is onto (otherwise, take $K = \text{Im}(f)$). Define $h: G/N \rightarrow H$ by



Well-defined: If $xN = yN$, then $y^{-1}xN = N$, so $y^{-1}x \in N = \text{Ker}(\pi) \leq \text{Ker}(f)$. Now,

$$f(y^{-1}x) = 1 \implies f(y)^{-1}f(x) = 1 \implies h(xN) = f(x) = f(y) = h(yN). \quad \checkmark$$

Homomorphism: $h(xNyN) = h(xyN) = f(xy) = f(x)f(y) = h(xN)h(yN). \quad \checkmark$

Uniqueness: Follows from existence, since f and π are quotients (cancellation laws). \checkmark

The co-universal property of quotient groups

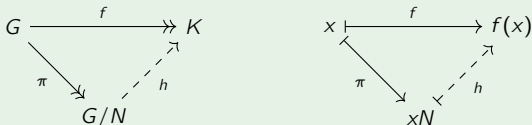
Theorem

Let $N \trianglelefteq G$ and $f: G \rightarrow K$ be a homomorphism such that $N \leq \text{Ker}(f)$. Then

1. f uniquely factors through $\pi: G \rightarrow G/N$ (i.e., $\exists! h: G/N \rightarrow K$ such that $g = h \circ \pi$).
2. h is injective iff $\text{Ker}(f) = N$.

Proof (ii)

Assume WLOG that f is onto. We just found the unique h such that



Let $H = \text{Ker}(f)$, and note that

$$\text{Ker}(h) = \{xN \mid f(x) = 1_K\} = \{xN \mid x \in H\} = H/N.$$

Note that h is injective iff $|\text{Ker}(h)| = 1$, or equivalently, $H = N$. □

Co-universal property of quotient groups \Rightarrow FHT

Corollary: Fundamental homomorphism theorem

If $f: G \rightarrow H$ is a homomorphism, then $G/\text{Ker}(f) \cong \text{Im}(f)$.

Proof

Let $K = \text{Im}(f)$ and $N = \text{Ker}(f)$ with canonical quotient map $\pi: G \rightarrow G/N$.

By construction, $\text{Ker}(f) = N = \text{Ker}(\pi)$.

By the co-universal property of quotient maps, f factors through the quotient:

$$\begin{array}{ccc} G & \xrightarrow{f} & \text{Im}(f) \\ \pi \searrow & & \nearrow \iota \\ & G/N & \end{array} \qquad \begin{array}{ccc} x & \xrightarrow{f} & f(x) \\ \pi \searrow & & \nearrow \iota \\ & xN & \end{array}$$

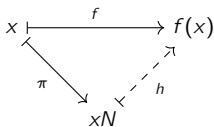
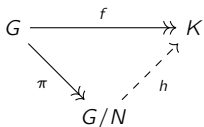
Since $\text{Ker}(f) = N$, the map ι is injective by Part (ii) of the previous theorem.

Therefore, ι is an isomorphism. □

Abstracting the (co)-universal property

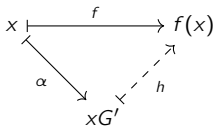
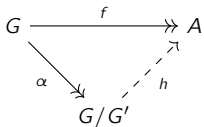
To motivate where we're going, let's rephrase what we just did as

" G/N is the *largest quotient that collapses N* , in that any other homomorphism collapsing N *factors through $\pi: G \rightarrow G/N$ uniquely.*"



Compare this to what we know about the commutator subgroup G' :

" G/G' is the *largest abelian quotient* of G , in that any other homomorphism to an abelian group *factors through $\alpha: G \rightarrow G/G'$ uniquely.*"



Abstracting the (co)-universal property

The **co-universal property** of quotients came with a distinguished (maximal)

- **group** G/N , and
- canonical **map** $\pi: G \rightarrow G/N$.

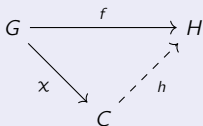
Definition

A **co-universal pair** (C, χ) for a group G w.r.t. a property consists of:

- a **group** C , with
- an **incoming map** $\chi: G \rightarrow C$,

such that every $f: G \rightarrow H$ with the same property factors through χ uniquely.

I.e., there is a unique homomorphism $h: C \rightarrow H$ between **co-domains** such that $f = h \circ \chi$.



Abstracting the (co)-universal property

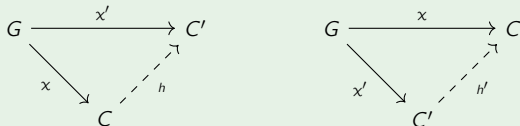
Proposition

If G has a co-universal pair (C, χ) w.r.t. some property, C is unique up to isomorphism.

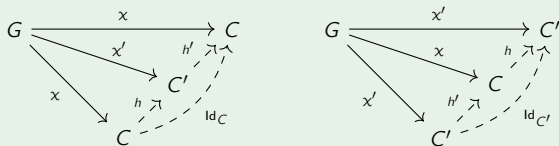
Proof

Let (C, χ) and (C', χ') be co-universal. Start with (C, χ) , and take $H = C'$ and $f = \chi'$.

By definition, $\exists! h: C \rightarrow C'$ such that $\chi' = h \circ \chi$. Reverse the roles, and we get:



We can “stack” one diagrams on the other, and vice-versa:



By uniqueness, $h \circ h' = \text{Id}_C$ (left), and $h' \circ h = \text{Id}_{C'}$ (right). Thus, $C \cong C'$. □

A co-universal property and nilpotency

Recall that we characterized nilpotent groups via iterative “maximal central descents.”

Given $N \trianglelefteq G$, the **maximal central descent** $[G, N]$ is characterized as being

“the smallest subgroup L such that N/L is central in G/L ”.

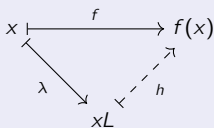
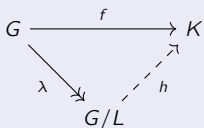
We can phrase this as a co-universal property.

Consider (L, λ) , where $L = [G, N]$ and $\lambda: G \rightarrow G/L$ is the canonical quotient.

Co-universal property of central descents (HW)

Let $N \trianglelefteq G$ and $f: G \rightarrow K$ for which $f(N)$ is central. Then f **uniquely factors through the canonical quotient map** $\lambda: G \rightarrow G/L$, where $L = [G, N]$.

That is, there is a unique homomorphism $h: G/L \rightarrow K$ for which $f = h \circ \lambda$.



Universal vs. co-universal properties

We call the examples we've seen **co-universal** because the map is between the **co-domains**.

The "dual" version, where the maps is between the domains, are **universal properties**.

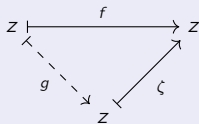
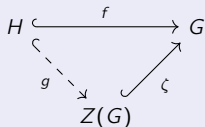
Most books don't distinguish these two, and use "universal" for both.

The examples we've seen were **maximal quotients**. Let's now look at **maximal subgroups**.

Universal property of centers

Let $H \leq G$ for which $xz = zx$ for all $z \in H$ and $x \in G$. The canonical inclusion $g: H \hookrightarrow G$ uniquely factors through $\zeta: Z(G) \hookrightarrow G$.

That is, there is a unique embedding $f: H \hookrightarrow Z(G)$ for which $f = \zeta \circ g$.

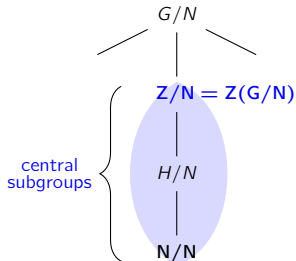
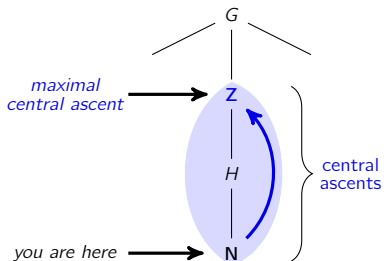
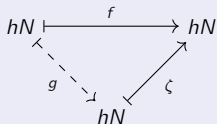
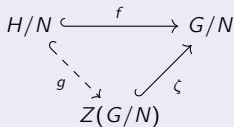


Another universal property

Universal property of central ascents

Given $N \trianglelefteq G$, suppose that $H/N \leq Z(G/N)$. The canonical inclusion $H/N \hookrightarrow G/N$ uniquely factors through $\zeta: Z(G/N) \hookrightarrow G/N$.

That is, there is a unique embedding $g: H/N \hookrightarrow Z(G/N)$ for which $f = \zeta \circ g$.

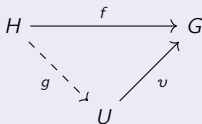


Universal pairs and universal constructions

Definition

A **universal pair** (U, ν) for G w.r.t. a property consists of a group U and map $\nu: U \rightarrow G$, such that every other $f: H \rightarrow G$ with the same property factors through ν uniquely.

That is, $\exists! g: H \rightarrow U$ between the *domains* such that $f = \nu \circ g$.



Proposition (HW)

If G has a universal pair (U, ν) w.r.t. some property, then U is unique up to isomorphism.

It's not standard or necessary to characterize a simple concept like $Z(G)$ with a universal property. We did it as a "warm up."

Soon, we'll *define* concepts by a (co-)universal property.

These are examples of **universal constructions**.

Motivation: direct product vs. direct sums

Open-ended question

What is the limit of $\mathbb{R}^n = \{(x_1, \dots, x_n) \mid x_i \in \mathbb{R}\}$, as $n \rightarrow \infty$?

- Define \mathbb{R}^∞ to be the space of all **infinite sequences**

$$\mathbb{R}^\infty := \prod_{i=1}^{\infty} \mathbb{R} := \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \cdots = \{(a_1, a_2, a_3, \dots) \mid a_i \in \mathbb{R}\}.$$

This space contains “vectors” such as $(1, 1, 1, \dots)$. We’ll call it the “**direct product**.”

- Define \mathbb{E}^∞ to be the space of all **finite sums**, like

$$\mathbf{e} = a_1 \mathbf{e}_1 + \cdots + a_n \mathbf{e}_n = \sum_{i=1}^n a_i \mathbf{e}_i, \quad \|\mathbf{v}\| = \sqrt{a_1^2 + \cdots + a_n^2}.$$

We’ll call this the “**direct sum**”.

$$\begin{aligned} \mathbb{E}^\infty &:= \bigoplus_{i=1}^{\infty} \mathbb{R} \mathbf{e}_i := \mathbb{R} \mathbf{e}_1 \oplus \mathbb{R} \mathbf{e}_2 \oplus \mathbb{R} \mathbf{e}_3 \oplus \cdots = \left\{ \sum_{i=1}^k a_i \mathbf{e}_i \mid a_i \in \mathbb{R}, k \geq 1 \right\} \\ &\cong \{(a_1, a_2, a_3, \dots) \mid a_i \in \mathbb{R}, \text{ all but finitely many } a_j \text{ are zero}\}. \end{aligned}$$

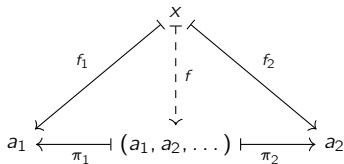
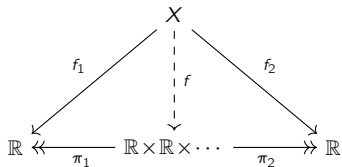
Motivation: direct product vs. direct sums

Define the canonical quotient maps for each $i = 1, 2, \dots$ as

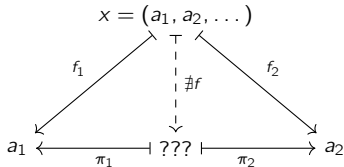
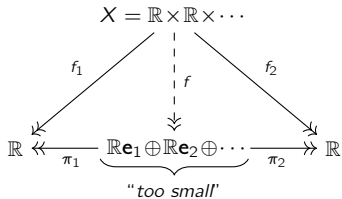
$$\pi_i: \mathbb{R} \times \mathbb{R} \times \dots \longrightarrow \mathbb{R}, \quad \pi_i: (a_1, a_2, \dots) \longmapsto a_i.$$

The direct product is the “*smallest P that projects onto each factor.*”

Given any family $f_i: X \rightarrow \mathbb{R}$ of maps, each f_i factors through the projection $\pi_i: P \rightarrow \mathbb{R}$.



Let's see why this fails if we tried to use \mathbb{E}^∞ for P :



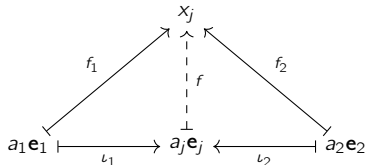
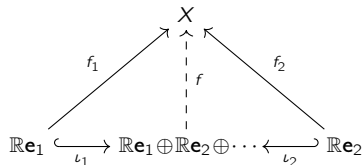
Motivation: direct product vs. direct sums

Define the **natural inclusion map** for each $j = 1, 2, \dots$ as

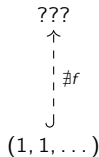
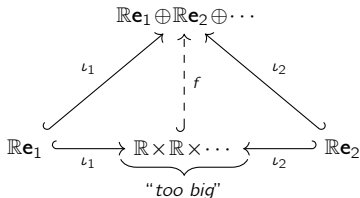
$$\iota_j: \mathbb{R}e_j \hookrightarrow \bigoplus_{i=1}^{\infty} \mathbb{R}e_i, \quad \iota_j: a_j e_j \mapsto a_j e_j$$

The direct sum is the "**smallest S that each factor embeds into.**"

Given any family $f_j: \mathbb{R}e_j \rightarrow X$ of maps, each ι_j factors through the embedding $\iota_j: \mathbb{R}e_j \hookrightarrow S$.

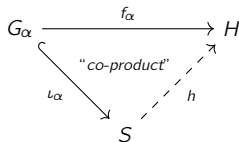
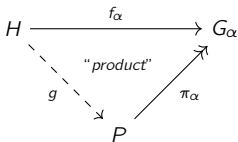


Let's see why this fails if we try to use \mathbb{R}^{∞} for S :



Returning to groups

Let $\{G_\alpha \mid \alpha \in A\}$ be a nonempty family of groups. We will define their product and co-product via a **universal construction**.



Remark

Existence of the map needed to make these diagrams commute does *not* imply uniqueness from the cancellation laws – each is the “wrong type” of diagram for that.

The fact that there are such groups that guarantee uniqueness indicates that the definitions are capturing something fundamentally important.

Definition

The **product** of $\{G_\alpha \mid \alpha \in A\}$ is a group P with a family of homomorphisms $\{\pi_\alpha : P \rightarrow G_\alpha \mid \alpha \in A\}$, satisfying:

Given any group H and homomorphisms $f_\alpha : H \rightarrow G_\alpha$, there is a unique homomorphism $g : H \rightarrow P$ such that $\pi_\alpha \circ g = f_\alpha$ for all $\alpha \in A$.

Products: surjectivity and uniqueness

Proposition

If $\{G_\alpha \mid \alpha \in A\}$ has a product, it is unique up to isomorphism, and each π_α is surjective.

Proof

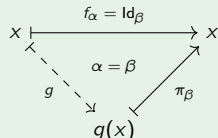
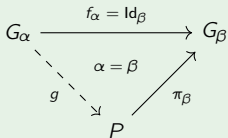
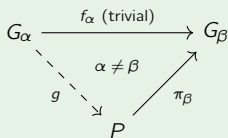
We've shown uniqueness.

To show that π_α is surjective, consider $\pi_\beta: P \rightarrow G_\beta$, and take $H = G_\alpha$.

Define f_β to be the identity map if $\beta = \alpha$ and the trivial map otherwise. That is,

$$f_\alpha: G_\alpha \longrightarrow G_\beta, \quad f_\alpha(x) = \begin{cases} x, & \alpha = \beta \\ 1, & \alpha \neq \beta. \end{cases}$$

Every element $x \in G_\beta$ has a π_β -preimage, $g(x) \in P$.



Products: existence

Proposition

The product of $\{G_\alpha \mid \alpha \in A\}$ is the Cartesian product, $P = \prod_{\alpha \in A} G_\alpha$.

Proof

Define the canonical projection maps as

$$\pi_\beta: P \longrightarrow G_\beta, \quad \pi_\beta: (x_\alpha)_{\alpha \in A} \longmapsto x_\beta.$$

Suppose we have another family of maps $f_\alpha: H \rightarrow G_\alpha$, for each $\alpha \in A$.

Goal. Show $\exists! g: H \rightarrow P$ such that $f_\alpha = \pi_\alpha \circ g$ for all $\alpha \in A$.

$$\begin{array}{ccc} H & \xrightarrow{f_\beta} & G_\beta \\ \text{---} \swarrow \text{---} & & \nearrow \text{---} \\ & \prod_{\alpha \in A} G_\alpha & \\ & \text{---} \swarrow \text{---} & \nearrow \text{---} \\ & & \end{array} \quad \begin{array}{ccc} x & \xrightarrow{f_\beta} & f_\beta(x) \\ \text{---} \swarrow \text{---} & & \nearrow \text{---} \\ & \prod_{\alpha \in A} (f_\alpha(x)) & \\ & \text{---} \swarrow \text{---} & \nearrow \text{---} \\ & & \end{array}$$

Uniqueness. Suppose $\exists h: H \rightarrow P$ for which $f_\alpha = \pi_\alpha \circ h$ for all $\alpha \in A$.

This means $\pi_\alpha \circ g = \pi_\alpha \circ h$. Take $x \in H$, note that

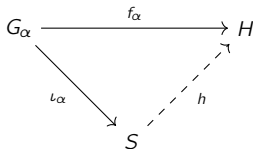
$$h(x)_\beta = \pi_\beta(h(x)) = f_\beta(x) = \pi_\beta(g(x)) = g(x)_\beta. \quad \square$$

Co-products

Definition

The **co-product** of $\{G_\alpha \mid \alpha \in A\}$ is a group S with a family of homomorphisms $\{\iota_\alpha: G_\alpha \rightarrow S \mid \alpha \in A\}$, satisfying:

Given any group H and homomorphisms $f_\alpha: G_\alpha \rightarrow H$, there is a unique homomorphism $h: S \rightarrow H$ such that $h \circ \iota_\alpha = f_\alpha$ for all $\alpha \in A$.



Exercise (HW)

If $\{G_\alpha \mid \alpha \in A\}$ has a co-product, it is unique up to isomorphism, and each ι_α is injective.

Showing existence of a co-product is trickier – it is a construction that we have not yet seen.

The product of C_2 and C_2 has order 4. The co-product is infinite.

Categories

Some constructions we've recently seen have analogues for other mathematical objects.

We can define the product and coproduct of sets, topological spaces, rings, vector spaces, etc.

Many structural results carry over, so we'd like to generalize these in a common framework.

The mathematical field that addresses these questions is called **category theory**.

Definition

A **category** \mathcal{C} consists of

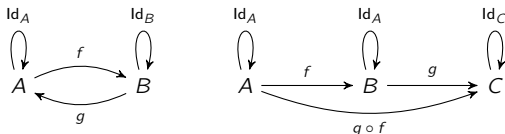
- a class $\text{Ob}(\mathcal{C})$ of **objects**,
 - a class $\text{Hom}(\mathcal{C})$ of **morphisms** between objects, with identities, closure, and associativity.
-
- Examples of "objects" include sets, groups, rings, vector spaces, topological spaces, etc.,
 - "Morphisms" are meant to be "structure-preserving maps."

Categories

Think of the category $\mathcal{C} = \mathbf{Grp}$ of groups as a massive directed multigraph, where

- each node represents a group
- there is a directed edge from A to B for each homomorphism $f: A \rightarrow B$.

We require: **identity**, **composition**, and **associativity**.



Denote the morphisms from A to B by $\text{Hom}_{\mathcal{C}}(A, B)$.

- (i) Every group has an **identity morphism**: for every $A \in \text{Ob}(\mathcal{C})$, there is $\text{Id}_A \in \text{Hom}_{\mathcal{C}}(A, A)$ satisfying

$$f \circ \text{Id}_A = f, \text{ for all } f \in \text{Hom}_{\mathcal{C}}(A, B), \quad \text{Id}_B \circ g = g, \text{ for all } g \in \text{Hom}_{\mathcal{C}}(A, B).$$

- (ii) Morphisms are **closed under composition**:

$$\text{If } f \in \text{Hom}_{\mathcal{C}}(A, B) \text{ and } g \in \text{Hom}_{\mathcal{C}}(B, C), \text{ then } g \circ f \in \text{Hom}_{\mathcal{C}}(A, C).$$

- (iii) Composition of morphisms is **associative**:

$$\text{If } f \in \text{Hom}_{\mathcal{C}}(A, B), \text{ } g \in \text{Hom}_{\mathcal{C}}(B, C), \text{ } h \in \text{Hom}_{\mathcal{C}}(C, D), \text{ then } h \circ (g \circ f) = (h \circ g) \circ f.$$

Abstracting the notion of “one-to-one” and “onto”

Definition

Let $f, f_1, f_2 \in \text{Hom}_{\mathcal{C}}(A, B)$ and $g, g_1, g_2 \in \text{Hom}_{\mathcal{C}}(B, C)$. Then

1. g is a **monomorphism** if $g \circ f_1 = g \circ f_2$ implies $f_1 = f_2$.
2. f is an **epimorphism** if $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$.

Sometimes, we'll say “*mono*” and “*epi*” (nouns) or *monic* and “*epic*” (adjectives).

A morphism $f \in \text{Hom}_{\mathcal{C}}(A, B)$ is an **isomorphism** if it has a two-sided inverse.

That is, if $\exists g \in \text{Hom}_{\mathcal{C}}(B, A)$ such that $g \circ f = \text{Id}_A$ and $f \circ g = \text{Id}_B$.

We say A and B are **equivalent**.

$$A \begin{array}{c} \xrightarrow{f_1} \\ \xrightarrow{f_2} \end{array} B \xrightarrow{g} C$$

“*monomorphism*” (one-to-one)

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{g_1} \\ \xrightarrow{g_2} \end{array} C$$

“*epimorphism*” (onto)

$$\begin{array}{ccc} \text{Id}_A = g \circ f & & \text{Id}_B = f \circ g \\ \downarrow & & \downarrow \\ A & \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{g} \end{array} & B \\ & & \downarrow \\ & & \text{Id}_B \end{array}$$

“*equivalent*” (isomorphic)

Abstracting the notion of “product” and “coproduct”

Definition

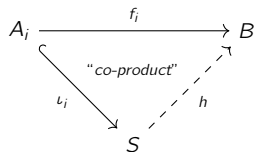
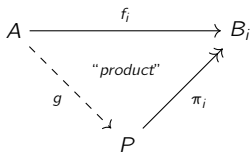
Consider a category \mathcal{C} and a non-empty collection $\{B_i \mid i \in I\}$ of objects.

A **product** for $\{B_i\}$ is $P \in \text{Ob}(\mathcal{C})$ with a family $\{\pi_i \in \text{Hom}(P, B_i) \mid i \in I\}$ such that:

Given any $A \in \text{Ob}(\mathcal{C})$ and $\{f_i \in \text{Hom}_{\mathcal{C}}(A, B_i) \mid i \in I\}$, there is a unique $g \in \text{Hom}_{\mathcal{C}}(A, P)$ such that $\pi_i \circ g = f_i$ for all $i \in I$.

A **coproduct** for $\{B_i\}$ is $S \in \text{Ob}(\mathcal{C})$ with $\{\iota_i \in \text{Hom}(A_i, S) \mid i \in I\}$ such that:

Given any $B \in \text{Ob}(\mathcal{C})$ and family $\{f_i \in \text{Hom}_{\mathcal{C}}(A_i, B) \mid i \in I\}$, there is a unique $h \in \text{Hom}_{\mathcal{C}}(S, B)$ such that $h \circ \iota_i = f_i$ for all $i \in I$.



(1)

It can be shown that the π_i 's are epimorphisms, and ι_i 's are monomorphisms.

A few counterintuitive facts

- *Isomorphisms \neq mono + epi!* In the category **Rng**, the non-surjective morphism

$$g: \mathbb{Z} \longrightarrow \mathbb{Q}, \quad g(n) = n$$

is both monic and epic.

$$R \begin{array}{c} \xrightarrow{g_1} \\ \xrightarrow{g_2} \end{array} \mathbb{Z} \xrightarrow{f} \mathbb{Q} \qquad \mathbb{Z} \xrightarrow{g} \mathbb{Q} \begin{array}{c} \xrightarrow{h_1} \\ \xrightarrow{h_2} \end{array} \mathbb{R}$$

The equality $f \circ g_1 = f \circ g_2$, implies $g_1 = g_2$, and $h_1 \circ g = h_2 \circ g$, forces $h_1 = h_2$.

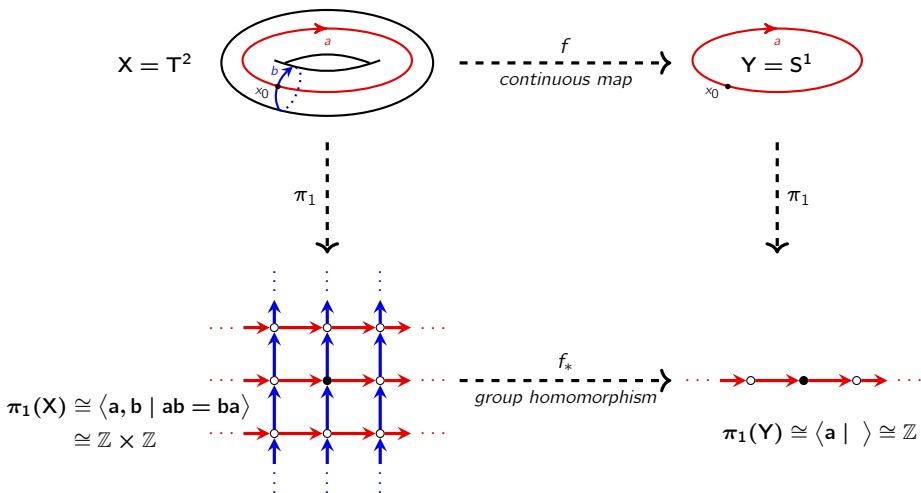
However, g is not an isomorphism because it does not have a left or a right inverse.

- *The same concept across different categories can seem very different!*

Category	Objects	Morphisms	Product	coproduct
Set	sets	functions	Cartesian product	disjoint union
Grp	groups	homomorphisms	direct product	free product
Ab	abelian groups	homomorphisms	direct product	direct sum
Ring	rings w/ 1	ring homomorphisms	direct product	free product
Field	fields	field embeddings	none	none
Vect$_{\mathbb{F}}$	\mathbb{F} -vector spaces	linear functions	direct product	direct sum
Top	topological spaces	continuous maps	product topology	disjoint union

A functor from **Top** to **Grp**

Sometimes, there are structure-preserving maps between categories.



This is an example of a **functor**.

A functor from **Top** to **Grp**

The **fundamental group** of X is the group $\pi_1(X)$ of all “loops up to equivalence.”

A continuous map $f: X \rightarrow Y$ induces a homomorphism

$$f_*: \pi_1(T^2) \longrightarrow \pi_1(S^1), \quad f_*: (a, b) \longmapsto a.$$

Formally, π_1 is a functor from **Top**_• to **Grp**, defined as:

$$\begin{aligned} \pi_1: \text{Ob}(\mathbf{Top}_\bullet) &\longrightarrow \text{Ob}(\mathbf{Grp}) & \mathcal{F}: \text{Hom}(\mathbf{Top}_\bullet) &\longrightarrow \text{Hom}(\mathbf{Grp}) \\ X &\longmapsto \pi_1(X) & X \xrightarrow{f} Y &\longmapsto \pi_1(X) \xrightarrow{f_*} \pi_1(Y) \end{aligned}$$

For arbitrary (pointed) topological spaces (X, x_0) and (Y, y_0) :

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \pi_1 \downarrow & & \downarrow \pi_1 \\ \pi_1(X) & \xrightarrow{f_*} & \pi_1(Y) \end{array}$$

f is a contin. b/w topological spaces, with $f(x_0) = y_0$

f_* is a homomorphism b/w fundamental groups

Covariant and contravariant functors

Definition

A (**covariant**) **functor** \mathcal{F} from \mathcal{C} to \mathcal{D} is a function that sends

- objects A of \mathcal{C} to objects $\mathcal{F}(A)$ of \mathcal{D} ,
- morphisms $f: A \rightarrow B$ in \mathcal{C} to morphisms $\mathcal{F}(f): \mathcal{F}(A) \rightarrow \mathcal{F}(B)$ in \mathcal{D} satisfying:
 - $\mathcal{F}(\text{Id}_A) = \text{Id}_{\mathcal{F}(A)}$ for all $A \in \text{Ob}(\mathcal{C})$
 - $\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$ for all morphisms $f: A \rightarrow B$ and $g: B \rightarrow C$.

$$\begin{array}{ccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ & \searrow & & \nearrow & \\ & & g \circ f & & \end{array} \qquad \begin{array}{ccc} \mathcal{F}(A) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}(B) & \xrightarrow{\mathcal{F}(g)} & \mathcal{F}(C) \\ & \searrow & & \nearrow & \\ & & \mathcal{F}(g) \circ \mathcal{F}(f) & & \end{array}$$

There is a “dual” type of functor, called **contravariant**, that reverses the arrows.

That is, they send $A \xrightarrow{f} B$ to $\mathcal{F}(B) \xrightarrow{\mathcal{F}(f)} \mathcal{F}(A)$

$$\begin{array}{ccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ & \searrow & & \nearrow & \\ & & g \circ f & & \end{array} \qquad \begin{array}{ccc} \mathcal{F}(A) & \xleftarrow{\mathcal{F}(f)} & \mathcal{F}(B) & \xleftarrow{\mathcal{F}(g)} & \mathcal{F}(C) \\ & \searrow & & \nearrow & \\ & & \mathcal{F}(g) \circ \mathcal{F}(f) & & \end{array}$$

A contravariant functor from linear algebra

Let $V \in \text{Ob}(\mathbf{Vect}_{\mathbb{R}})$ be an n -dimensional vector space.

The **dual space** $V^* \in \text{Ob}(\mathbf{Vect}_{\mathbb{R}})$ consists of all *linear scalar functions* $\ell: V \rightarrow \mathbb{R}$.

Think of:

- elements in V as column vectors,
- elements in V^* as row vectors.

$$\ell: V \longrightarrow \mathbb{R}, \quad \ell(v) = \underbrace{[a_1 \quad a_2 \quad \cdots \quad a_n]}_{\ell \in V^*} \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{v \in V} = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n.$$

A linear map $A: V \rightarrow W$ can be represented by an $m \times n$ matrix, where $\dim(W) = m$.

Think of this as *left-multiplication by column vectors*, $Av = w$:

$$A: V \longrightarrow W, \quad \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{A \in \text{Hom}(V, W)} \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{v \in V} = \underbrace{\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix}}_{w \in W}.$$

A contravariant functor from linear algebra

The *transpose* is a linear map $A^t: W^* \rightarrow V^*$.

Think of this as *right-multiplication by row vectors*, $w^t A^t = v^t$:

$$A^t: W^* \longrightarrow V^*, \quad \underbrace{[w_1 \quad w_2 \quad \cdots \quad w_m]}_{w^t \in W^*} \underbrace{\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1m} & a_{2m} & \cdots & a_{nm} \end{bmatrix}}_{A^t \in \text{Hom}(W^*, V^*)} = \underbrace{[v_1 \quad v_2 \quad \cdots \quad v_n]}_{v^t \in V^*}$$

Formally, we have a contravariant functor:

$$\begin{aligned} \mathcal{F}: \text{Ob}(\mathbf{Vect}_{\mathbb{F}}) &\longrightarrow \text{Ob}(\mathbf{Vect}_{\mathbb{F}}) & \mathcal{F}: \text{Hom}(\mathbf{Vect}_{\mathbb{F}}) &\longrightarrow \text{Hom}(\mathbf{Vect}_{\mathbb{F}}) \\ V &\longmapsto V^* & v \xrightarrow{A} w &\longmapsto w^t \xrightarrow{A^*} v^t \end{aligned}$$

Notice how the arrow on the bottom of the following commutative diagram is reversed; this is contravariance.

$$\begin{array}{ccc} V & \xrightarrow{A} & W \\ \mathcal{F} \downarrow & & \downarrow \mathcal{F} \\ V^* & \xleftarrow{A^*} & W^* \end{array} \quad \begin{aligned} A &\in \text{Hom}(V, W) \\ A^* &\in \text{Hom}(W^*, V^*) \end{aligned}$$

Abelianization, as a functor from **Grp** to **Ab**

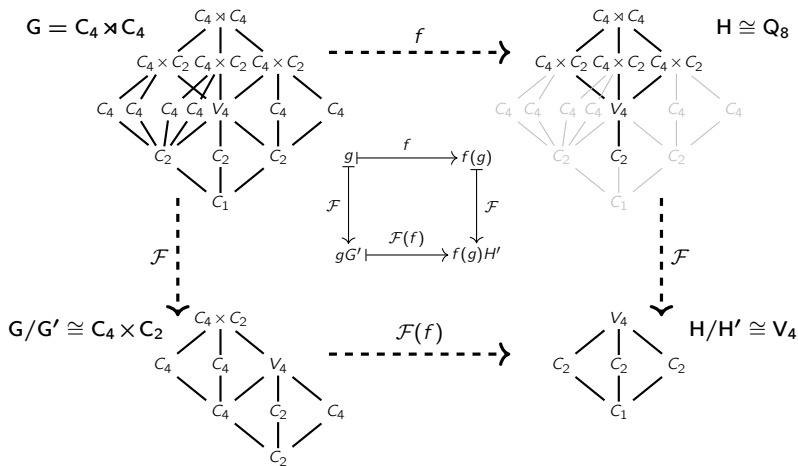
Consider the functor sending a group G to its abelianization $A \cong G/G' = G/[G, G]$:

$$\mathcal{F}: \text{Ob}(\mathbf{Grp}) \longrightarrow \text{Ob}(\mathbf{Ab})$$

$$\mathcal{F}: \text{Hom}(\mathbf{Grp}) \longrightarrow \text{Hom}(\mathbf{Ab})$$

$$G \longmapsto G/G'$$

$$g \mapsto f(g) \longmapsto gG' \mapsto f(g)H'$$



Initial and terminal objects

Definition

An object $I \in \text{Ob}(\mathcal{C})$ is **initial** if for each $C_i \in \text{Ob}(\mathcal{C})$, there is a unique $\pi_i \in \text{Hom}_{\mathcal{C}}(I, C_i)$.

An object $T \in \text{Ob}(\mathcal{C})$ is **terminal** if for each $C_i \in \text{Ob}(\mathcal{C})$, there is a unique $\iota_i \in \text{Hom}_{\mathcal{C}}(C_i, T)$.

An object that is initial and terminal is called a **zero object**.

Sometimes, initial objects are called *universal* or *coterminal*, and terminal objects are *final* or *couniversal*.

Category	Objects	Initial objects	Terminal objects	Zero objects
Set	sets	\emptyset	every $\{x\}$	none
Grp	groups	$\langle e \rangle$	$\langle e \rangle$	$\langle e \rangle$
Ab	abelian groups	$\langle 0 \rangle$	$\langle 0 \rangle$	$\langle 0 \rangle$
Rng	rings	$\{0\}$	$\{0\}$	$\{0\}$
Ring	rings w/ 1	\mathbb{Z}	$\{0\}$	none
Field	fields	none	none	none
Field _p	fields w/ char. $p > 0$	\mathbb{Z}_p	none	none
Vect _F	\mathbb{F} -vector spaces	$\{0\}$	$\{0\}$	$\{0\}$
Top	topological spaces	\emptyset	every $\{x\}$	none

Initial and terminal objects

Proposition

Any two initial objects in a category \mathcal{C} are equivalent.

Proof

Let I and J be initial.

Since I is initial, there is a unique morphism $f \in \text{Hom}_{\mathcal{C}}(I, J)$.

Since J is initial, there is a unique morphism $g \in \text{Hom}_{\mathcal{C}}(J, I)$.

The morphism $g \circ f$ is in $\text{Hom}_{\mathcal{C}}(I, I)$, as is Id_I (below, left).

$$\begin{array}{ccc} I & \begin{array}{c} \xrightarrow{g \circ f} \\ \xrightarrow{\text{Id}_I} \end{array} & I \\ J & \begin{array}{c} \xrightarrow{f \circ g} \\ \xrightarrow{\text{Id}_J} \end{array} & J \end{array}$$

However, since I is initial, there must be a unique morphism in $\text{Hom}_{\mathcal{C}}(I, I)$, so $g \circ f = \text{Id}_I$.

Similarly, $f \circ g$ and Id_J are both in $\text{Hom}_{\mathcal{C}}(J, J)$ (above, right).

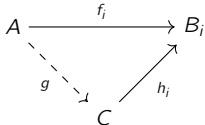
By uniqueness, $f \circ g = \text{Id}_J$, hence $I \cong J$. □

Uniqueness of products

Suppose $\{B_i \mid i \in I\}$ in \mathcal{C} has product P , with projections $\pi_i: P \rightarrow B_i$.

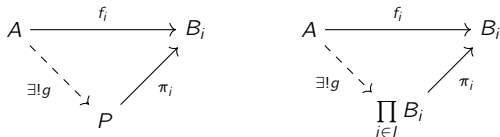
Define a new category \mathcal{B} :

- **objects**: families of maps $\{A \xrightarrow{f_i} B_i\}$
- **morphisms**: $A \xrightarrow{g} C$ that makes the following diagram commute.

$$g \in \text{Hom}_{\mathcal{B}}(A \xrightarrow{f_i} B_i, C \xrightarrow{h_i} B_i)$$


The diagram shows a triangle with vertices A , B_i , and C . A solid arrow labeled f_i points from A to B_i . A solid arrow labeled h_i points from C to B_i . A dashed arrow labeled g points from A to C .

A **terminal object** in \mathcal{B} is a family $\{P \xrightarrow{\pi_i} B_i\}$ such that for any $\{A \xrightarrow{f_i} B_i\}$, there exists a unique $g \in \text{Hom}_{\mathcal{C}}(A, P)$ that makes the diagram (left) commute:



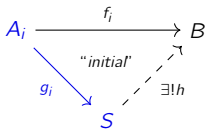
The left diagram shows a triangle with vertices A , B_i , and P . A solid arrow labeled f_i points from A to B_i . A solid arrow labeled π_i points from P to B_i . A dashed arrow labeled $\exists! g$ points from A to P .

The right diagram shows a triangle with vertices A , B_i , and $\prod_{i \in I} B_i$. A solid arrow labeled f_i points from A to B_i . A solid arrow labeled π_i points from $\prod_{i \in I} B_i$ to B_i . A dashed arrow labeled $\exists! g$ points from A to $\prod_{i \in I} B_i$.

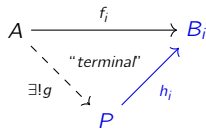
That is, the terminal object is the product! Thus, products are unique up to equivalence.

Uniqueness of coproducts and zero morphisms

We can construct an analogous category where the initial object is the coproduct.



$$h \in \text{Hom}_{\mathcal{A}} (A_i \xrightarrow{g_i} S, A_i \xrightarrow{f_i} B)$$

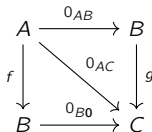


$$g \in \text{Hom}_{\mathcal{B}} (A \xrightarrow{f_i} B_i, P \xrightarrow{h_i} B_i)$$

Though each $\pi_i \in \text{Hom}_{\mathcal{C}}(P, B_i)$ need not be epic, there are conditions that guarantee this.

Definition

Let \mathcal{C} be a category with a zero object, $\mathbf{0} \in \text{Ob}(\mathcal{C})$. The **zero morphism** $0_{AB} \in \text{Hom}_{\mathcal{C}}(A, B)$ is the composition of the unique maps $A \rightarrow \mathbf{0} \rightarrow B$.



Zero morphisms

Proposition

If $\mathbf{0} \in \text{Ob}(\mathcal{C})$, the projection morphisms $\pi_i \in \text{Hom}_{\mathcal{C}}(P, B_i)$ of a product are epimorphisms.

Proof

Fix $\alpha \in I$, and define the family of maps $\{f_i \in \text{Hom}_{\mathcal{C}}(B_\alpha, B_i) \mid i \in I\}$ as

$$f_i: B_\alpha \longrightarrow B_i, \quad f_i = \begin{cases} \text{Id}_{B_\alpha}, & i = \alpha \\ 0_{B_i, \mathbf{0}}, & i \neq \alpha. \end{cases}$$

By the universal property of products, for each $i \in I$, we have:

A commutative diagram with B_α at the top left, B_i at the top right, and P at the bottom. A solid arrow points from B_α to B_i labeled $f_i = \text{Id}_{B_\alpha}$. A solid arrow points from P to B_i labeled $\pi_i = \pi_\alpha$. A dashed arrow points from B_α to P labeled $\exists! g_i = g_\alpha$. A solid arrow points from B_α to P labeled $i = \alpha$.

A commutative diagram with B_α at the top left, B_i at the top right, and P at the bottom. A solid arrow points from B_α to B_i labeled $f_i = 0_{B_i, \mathbf{0}}$. A solid arrow points from P to B_i labeled π_i . A dashed arrow points from B_α to P labeled $\exists! g_i$. A solid arrow points from B_α to P labeled $i \neq \alpha$.

To show π_α is epic, we need to verify left-cancellation.

Consider $f, g \in \text{Hom}_{\mathcal{C}}(B_\alpha, C)$ such that $f \circ \pi_\alpha = g \circ \pi_\alpha$.

Zero morphisms

Proposition

If $\mathbf{0} \in \text{Ob}(\mathcal{C})$, the projections $\pi_i \in \text{Hom}_{\mathcal{C}}(P, B_i)$ from a product are epimorphisms.

Proof

It suffices to show that $f = g$.

$$\begin{array}{ccccc} B_\alpha & \xrightarrow{f_\alpha = \text{Id}_{B_\alpha}} & B_\alpha & \xrightarrow[f]{g} & C \\ & \searrow g_\alpha & \nearrow \pi_\alpha & & \\ & & P & & \end{array}$$

By the commutativity of the diagram, we have

$$f = f \circ \text{Id}_{B_\alpha} = f \circ (\pi_\alpha \circ g_\alpha) = (f \circ \pi_\alpha) \circ g_\alpha = (g \circ \pi_\alpha) \circ g_\alpha = g \circ (\pi_\alpha \circ g_\alpha) = g \circ \text{Id}_{B_\alpha} = g,$$

whence π_α is an epimorphism. \square

Proposition (HW)

If $\mathbf{0} \in \text{Ob}(\mathcal{C})$, the inclusions $\iota_i \in \text{Hom}_{\mathcal{C}}(B_i, S)$ into a coproduct are monomorphisms.

Free groups

Throughout, let S be a nonempty set.

Definition

The **free group on S** is

$$F = F_S := \langle S \mid \rangle.$$

That is, F_S is generated by S , subject to no relations.

We can think of the free groups as groups where:

- elements are words in $T = S \sqcup S^{-1}$, where $S^{-1} := \{s^{-1} \mid s \in S\}$.
- the binary operation is concatenation.

The only way to modify words are by substitutions of form $ss^{-1} = 1$ and $s^{-1}s = 1$.

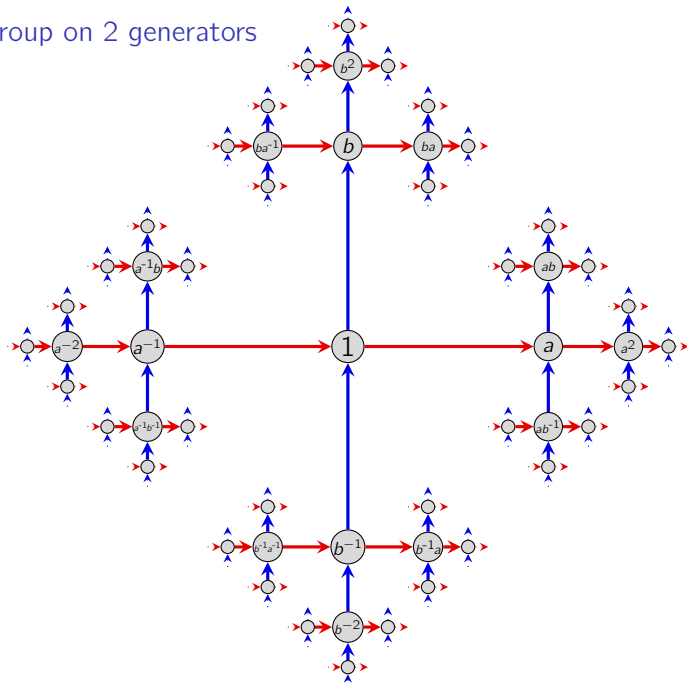
If $|S| = |T|$, then $F_S \cong F_T$.

If $|S| = n < \infty$, then $F_n := F_S$ is *free group on n generators*, or the *free group of rank n* .

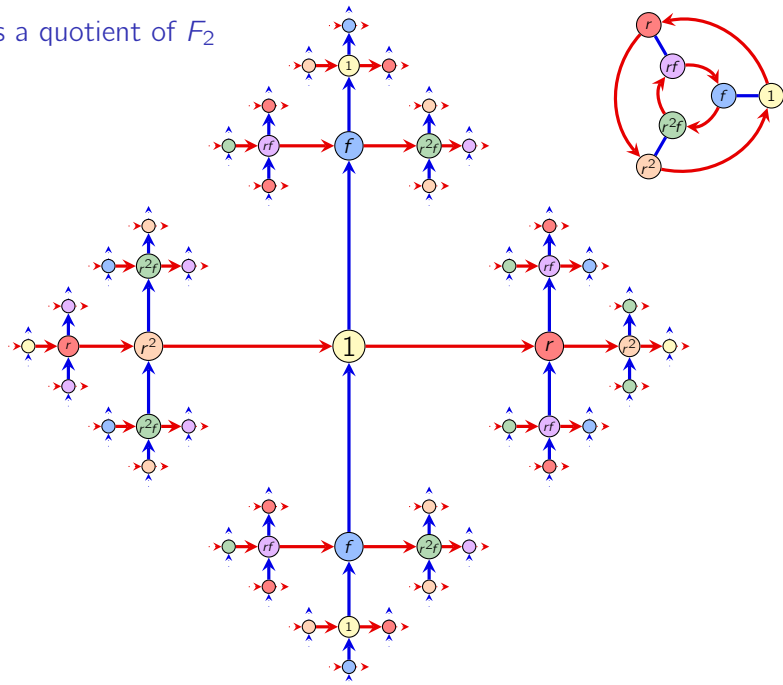
We'll soon see how every group is a quotient of a free group.

This can be formalized via a couniversal property.

The free group on 2 generators



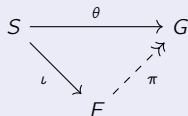
D_3 as a quotient of F_2



Free groups

Definition

A group F is **free** on $S \neq \emptyset$ if there is a function $\iota: S \rightarrow F$ such that for any other $\theta: S \rightarrow G$, there exists a unique homomorphism $\pi: F \rightarrow G$ such that $\theta = \pi \circ \iota$.



Proposition

If a free group exists on $S \neq \emptyset$, it is unique up to isomorphism, and $\iota: S \rightarrow F$ is injective.

Proof

We've seen uniqueness. Suppose ι is not 1-to-1; take $a \neq b$ in S for which $\iota(a) = \iota(b)$.

Consider the map $\theta: S \rightarrow \mathbb{Z}$,
$$\theta(s) = \begin{cases} 1 & s = a \\ 2 & s = b \\ 0 & s \notin \{a, b\}. \end{cases}$$

This forces $1 = \theta(a) = \pi(\iota(a)) = \pi(\iota(b)) = \theta(b) = 2$, a contradiction. □

Free semigroups

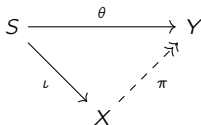
Definition

A **semigroup** is a set $X \neq \emptyset$ with associative binary operation.

A **homomorphism** is a function $f: X \rightarrow Y$ with $f(x_1x_2) = f(x_1)f(x_2)$ for all $x_1, x_2 \in X$.

Let **Sgp** denote the category of semigroups.

Free semigroups exist, are unique up to isomorphism, the map $\iota: S \rightarrow F$ is injective.



The free semigroup on $S = \{s\}$ is isomorphic to $\mathbb{N} = \{1, 2, \dots\}$ under addition.

Free semigroups

Proposition

If $S \neq \emptyset$, then there is a free semigroup over S .

Proof

Let X be the set of nonempty words over S , under concatenation:

$$X = S \cup (S \times S) \cup (S \times S \times S) \cup \cdots, \quad (a_1, \dots, a_n) * (b_1, \dots, b_m) = (a_1, \dots, a_n, b_1, \dots, b_m).$$

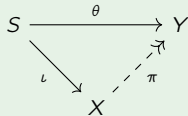
We'll show this is free over S , with inclusion map

$$\iota: S \longrightarrow X, \quad \iota(s) = s.$$

Given a function $\theta: S \rightarrow Y$ to another semigroup, define

$$\pi: X \longrightarrow Y, \quad \pi: (a_1, \dots, a_n) \longmapsto \theta(a_1) \cdots \theta(a_n).$$

Exercise. Check that π is a semigroup homomorphism, and $\pi \circ \iota = \theta$.



Free semigroups

Proposition

If $S \neq \emptyset$, then there is a free semigroup over S .

Proof (contin.)

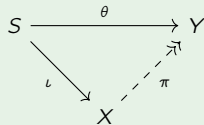
Given $\theta: S \rightarrow Y$, the function

$$\pi: X \longrightarrow Y, \quad \pi: (a_1, \dots, a_n) \longmapsto \theta(a_1) \cdots \theta(a_n).$$

satisfies $\pi \circ \iota = \theta$.

Uniqueness: Suppose $\sigma: X \rightarrow Y$ also satisfies $\sigma \circ \iota = \theta$. Then

$$\begin{aligned} \sigma((a_1, \dots, a_n)) &= \sigma(\iota(a_1) \cdots \iota(a_n)) \\ &= \sigma(\iota(a_1)) \cdots \sigma(\iota(a_n)) \\ &= \theta(a_1) \cdots \theta(a_n) \\ &= \pi(\iota(a_1)) \cdots \pi(\iota(a_n)) \\ &= \pi(\iota(a_1) \cdots \iota(a_n)) \\ &= \pi((a_1, \dots, a_n)). \end{aligned}$$



Therefore, X satisfies the co-universal property of free semigroups. □

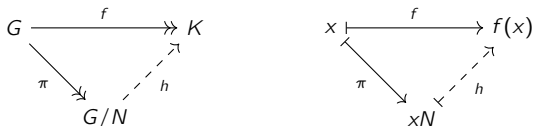
Quotient semigroups

Since semigroups lack an inverse, we don't have kernels, or isomorphism theorems.

But there is a co-universal property of quotient maps.

A group homomorphism $f: G \rightarrow K$ partitions G into cosets of $\text{Ker}(f)$.

If this is coarser than the partition of G into cosets of $N = \text{Ker}(\pi)$, then f factors through π :



A [relation](#) R on a semigroup Y is **well-defined** with respect to \cdot if

$$xRy \quad \text{and} \quad zRw \quad \implies \quad (x \cdot z)R(y \cdot w).$$

Let xR be the equivalence class containing x , and call

$$\pi: Y \longrightarrow Y/R, \quad \pi: y \longmapsto yR$$

the **canonical quotient map**.

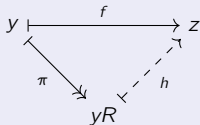
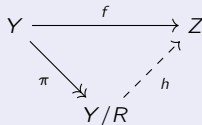
The **quotient semigroup** of Y is Y/R , with $xR \cdot yR := xyR$.

Co-universal property of quotient semigroups

Proposition

The quotient semigroup Y/R satisfies the following co-universal property:

If $f: Y \rightarrow Z$ is a semigroup homomorphism such that xRy implies $f(x) = f(y)$, then $\exists! h: Y/R \rightarrow Z$ such that $f = h \circ \pi$.



Proof

Existence follows from the definition: $h(yR) = h(\pi(y)) = f(y)$, with well-definedness automatic from $xRy \Rightarrow f(x) = f(y)$.

Uniqueness comes from the cancellation laws, because π is surjective. □

Construction of a free group over S

Given $S \neq \emptyset$, construct a disjoint set S' of “formal inverses”:

$$S' = \{s' \mid s \in S\}, \quad T = S \cup S'.$$

The bijection $s \mapsto s'$ and inverse $s' \mapsto s'' := s$ define a bijection $T \rightarrow T$, where $t \mapsto t'$.

Let X be the free semigroup on $T \subseteq X$ (under natural inclusion).

Call a homomorphism $\phi: X \rightarrow G$ **proper** if $\phi(s') = \phi(s)^{-1}$ for all $s \in S$.

If ϕ is proper, then $\phi(t') = \phi(t)^{-1}$ for all $t \in T$.

The only “relation” in a free group: $ss^{-1} = s^{-1}s = 1$ for all $s \in S \subseteq F$.

We'll construct this from the free semigroup by forcing $ss't = t$, for all $t \in T \subseteq X$.

If ϕ is proper, then

$$\phi(ss't) = \phi(s)\phi(s')\phi(t) = \phi(s)\phi(s)^{-1}\phi(t) = \phi(t).$$

Define an equivalence relation on X where

$$xRy \quad \text{iff} \quad \phi(x) = \phi(y) \text{ for every proper } \phi: X \rightarrow G.$$

Exercise: this is well-defined, and so X/R is a semigroup. .

Construction of a free group over S

We just showed that X/R is a semigroup. Now we'll show it's a group.

We'll write \bar{x} (not xR), so $\bar{x}\bar{y} = \overline{xy}$, and $\overline{x^{-1}} = \bar{x}^{-1}$.

Let $\pi: X \rightarrow X/R$ be the canonical quotient.

Identity. Choose any $s \in S$ and $x \in X$; we claim that $\overline{ss'} = 1$.

If $\phi: X \rightarrow G$ is proper, then $\phi(ss'x) = \phi(x)$, which means that $xRss'x$ in X , thus

$$\bar{x} = \overline{ss'x} = \overline{ss'} \cdot \bar{x}, \quad \text{and} \quad \bar{x} = \overline{xss'} = \bar{x} \cdot \overline{ss'}.$$

Thus, $\overline{ss'}$ is the identity. ✓

Inverses. Let $x = t_1 \cdots t_k \in X$.

We'll show that the inverse of \bar{x} is \bar{y} , where $y = t'_k \cdots t'_1$.

If ϕ is proper, then

$$\begin{aligned} \phi(xy) &= \phi(t_1 \cdots t_k t'_k \cdots t'_1) \\ &= \phi(t_1) \cdots \phi(t_k) \phi(t'_k) \cdots \phi(t'_1) \\ &= \phi(t_1) \cdots \phi(t_k) \phi(t_k)^{-1} \cdots \phi(t_1)^{-1} \\ &= 1_G = \phi(ss') \quad \text{for any } s \in S. \end{aligned}$$

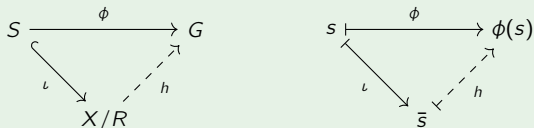
✓

Thus X/R is a group. □

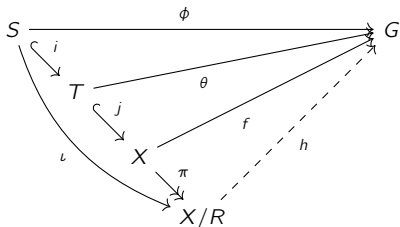
Showing that our free semigroup quotient X/R is free

Goal

Given $\iota: S \rightarrow X/R$ defined by $\iota(s) = \bar{s}$, show that for any map $\phi: S \rightarrow G$, there is a unique homomorphism $h: X/R \rightarrow G$ such that $\phi = h \circ \iota$

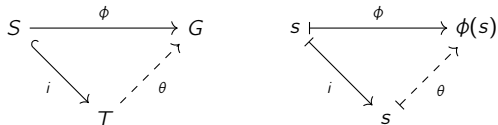


We'll build up this diagram in "pieces", culminating with the following:

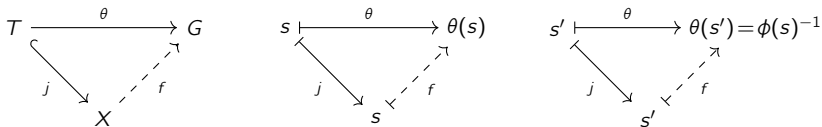


Showing that our free semigroup quotient X/R is free

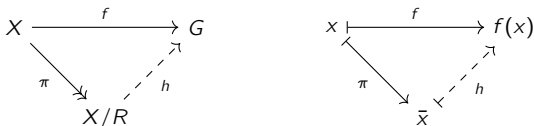
Extend $\phi: S \rightarrow G$ to a map $\theta: T \rightarrow G$ by setting $\theta(s') = \phi(s)^{-1}$.



Applying the co-universal property of free semigroups to $\theta: T \rightarrow G$ gives the following:

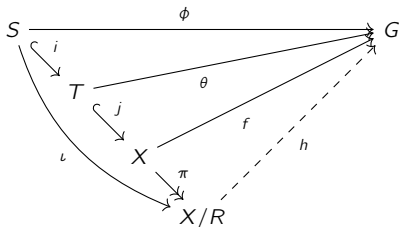


Since the homomorphism f is proper, the co-universal property of quotient semigroups gives:



Showing that our free semigroup quotient X/R is free

We know $\exists! h: X/R \rightarrow G$ such that $f = h \circ \pi$, but not necessarily $\phi = h \circ \iota$.



Suppose $\exists g: X/R \rightarrow G$ such that $\phi = g \circ \iota$. (Need $h = g$.)

We have $h \circ \pi \circ j \circ i = g \circ \pi \circ j \circ i$, and we claim that $h \circ \pi \circ j = g \circ \pi \circ j$.

It is clear that $h(\pi(j(s))) = f(\pi(j(s)))$ for all $s \in S$. By construction,

$$h(\pi(j(s'))) = h(\bar{s'}) = h(\bar{s}^{-1}) = h(s)^{-1} = g(s)^{-1} = g(\bar{s}^{-1}) = g(\bar{s'}) = g(\pi(j(s'))).$$

Therefore, $\theta = h \circ \pi \circ j = g \circ \pi \circ j$.

By the co-universal property of free semigroups, $\exists! f: X \rightarrow G$ such that $\theta = f \circ j$.

But both $h \circ \pi$ and $g \circ \pi$ satisfy this, and so $f = h \circ \pi = g \circ \pi \Rightarrow h = g$ □

Properties of free groups

Proposition

Suppose $S, U \neq \emptyset$. Then $F_S \cong F_U$ if and only if $|S| = |U|$.

Proof

“ \Rightarrow ” **Case 1:** $|S| < \infty$.

Each nonempty $R \subseteq S$ defines an index-2 subgroup, the kernel of

$$f_R: F_S \longrightarrow \mathbb{Z}_2, \quad f_R(s) = \begin{cases} 0 & s \in R \\ 1 & s \notin R \end{cases}$$

Since F_U has the same number of index-2 subgroups, $2^{|S|} - 1 = 2^{|U|} - 1 \Rightarrow |S| = |U|$.

Case 2: $|S| = \infty$.

Let $T = S \cup S^{-1}$. Then $|F_S| = |S|$ because.

$$|F_S| \leq 1 + |T| + |T \times T| + |T \times T \times T| + \cdots = \aleph_0 |T| = |S|.$$

Reversing roles gives $|F_U| = |U| = |S| = |F_S|$. ✓

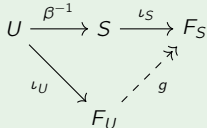
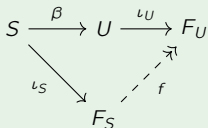
Properties of free groups

Proposition

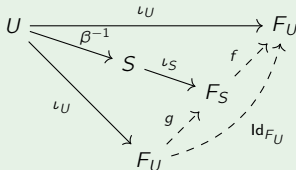
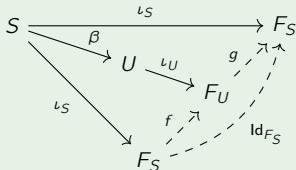
Suppose $S, U \neq \emptyset$. Then $F_S \cong F_U$ if and only if $|S| = |U|$.

Proof

" \Leftarrow " Fix a bijection $\beta: S \rightarrow U$ and use the co-universal property to get



We can "stack" these diagrams, two ways, to get:



By uniqueness, $g \circ f = \text{Id}_{F_S}$ and $f \circ g = \text{Id}_{F_U}$, so f and g are inverse isomorphisms. \square

Free objects

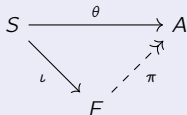
A functor $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{D}$ is **faithful** if $\mathcal{F}: \text{Hom}(\mathcal{C}) \rightarrow \text{Hom}(\mathcal{D})$ is injective.

A **concrete category** is a category \mathcal{C} with a faithful functor $\mathcal{F}: \mathcal{C} \rightarrow \mathbf{Set}$.

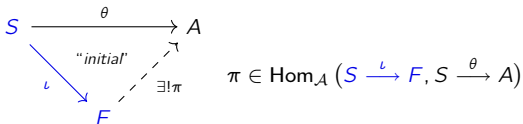
Definition

Let \mathcal{C} be concrete, $F \in \text{Ob}(\mathcal{C})$, and $\iota: S \rightarrow F$ a map of sets, where $S \neq \emptyset$.

Then F is **free on S** if for any $A \in \text{Ob}(\mathcal{C})$ and $\theta: S \rightarrow A$, there is a unique $\pi \in \text{Hom}_{\mathcal{C}}(F, A)$ such that $\iota \circ \pi = \theta$.



Like we did with products, we can construct a category where free objects are initial:



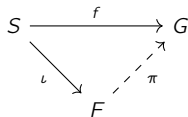
Free objects

Let \mathbf{Nil} be the category of nilpotent groups.

Suppose $\iota: S \rightarrow F$ is a free object in \mathbf{Nil} .

This means that every other nilpotent group G generated by S is a quotient of F :

“if G is nilpotent with set map $f: S \rightarrow G$, then there exists a unique $\pi: F \rightarrow G$ such that $f = \pi \circ \iota$.”



Suppose F has nilpotency class n . Then every quotient has nilpotency class $\leq n$. (Why?)

Thus, if $G = \langle S \rangle$ has nilpotency class $n + 1$, then $\nexists \pi: F \twoheadrightarrow G$.

Free objects

Let $\mathbf{Nil}_{\leq n}$ be the category of nilpotent groups of class $\leq n$.

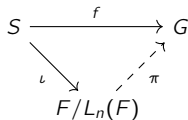
If G is a nilpotent group of class $\leq n$, then $L_n(G) = \langle 1 \rangle$.

$$\begin{aligned}L_1(G) &= [G, L_0] = [G, G] = \langle [g_1, g_0] \mid g_i \in G \rangle \\L_2(G) &= [G, L_1] = [G, [G, G]] = \langle [g_2, [g_1, g_0]] \mid g_i \in G \rangle \\L_3(G) &= [G, L_2] = [G, [G, [G, G]]] = \langle [g_3, [g_2, [g_1, g_0]]] \mid g_i \in G \rangle \\&\vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \\L_n(G) &= [G, L_{n-1}] = [G, [G, \dots [G, G]]] = \langle [g_n, [g_{n-1}, \dots [g_1, g_0]]] \mid g_i \in G \rangle\end{aligned}$$

Proposition

Let F be free on a set S . Then $F/L_n(F)$ is free in $\mathbf{Nil}_{\leq n}$.

“if G is nilpotent of class $\leq n$ and $f: S \rightarrow G$, then there exists a unique $\pi: F/L_n(F) \rightarrow G$ such that $f = \pi \circ \iota$.”

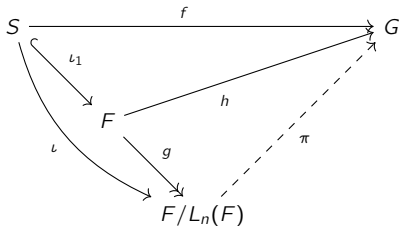


Free objects

Proposition

Let F be free on S . Then $F/L_n(F)$ is free in $\mathbf{Nil}_{\leq n}$.

The existence of $h: F \rightarrow G$ is because F is free on S .



Since G has nilpotent class $\leq n$, we have $\text{Ker}(g) = L_n(F) \leq \text{Ker}(h)$.

Now, π is guaranteed by the co-universal property of quotient maps.

Exercise: Verify that π is the unique map satisfying $f = \pi \circ \iota$.

Direct sums and bases

The **direct sum** of a family $\{A_i \mid i \in I\}$ of groups is

$$\bigoplus_{i \in I} A_i = \left\{ (a_i)_{i \in I} \in \prod_{i \in I} A_i \text{ with finite support} \right\}.$$

If all are abelian, let $\mathbf{e}_j := (a_i)_{i \in I}$ with $a_j = \delta_{ij}$. Every $x \in \bigoplus A_i$ can be written as

$$x = \sum_{i=1}^n a_i \mathbf{e}_i, \quad a_i \in \mathbb{Z}, \quad n \in \mathbb{N}.$$

If A is abelian, the subgroup generated by $X \subseteq S$ are the finite **linear combinations**:

$$\langle X \rangle = \{ a_1 x_1 + \cdots + a_n x_n \mid a_i \in \mathbb{Z}, x_i \in X \}.$$

A **basis** of A is a subset $X \subseteq A$ for which:

1. $A = \langle X \rangle$.
2. Given distinct $x_1, \dots, x_n \in X$,

$$a_1 x_1 + \cdots + a_n x_n = 0 \quad \implies \quad a_i = 0 \text{ for all } i = 1, \dots, n.$$

Direct sums and bases

Assuming the axiom of choice, in a vector space, Every generating set contains a basis.

This fails for abelian groups; e.g., $\mathbb{Z} = \langle 2, 3 \rangle$.

Every vector space has a basis, and every $v \neq 0$ is contained in one.

If an abelian group A has an element x of finite order, no basis can contain it.

Proposition

Let A be an abelian group with basis X . Then every $a \in A$ can be written as a unique (finite) linear combination of elements from X .

Proof

The following defines a homomorphism

$$f: \bigoplus_{i \in I} \mathbb{Z} \longrightarrow A, \quad f: \sum_{j=1}^n a_j \mathbf{e}_j \longmapsto \sum_{j=1}^n a_j x_j.$$

It is surjective by Property (1) of a basis, and has trivial kernel by Property (2).

Each way to write x as a linear combination of the basis elements corresponds to an f -preimage of x .

Uniqueness follows because f is bijective. □

Free abelian groups

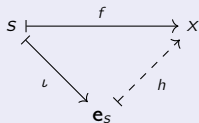
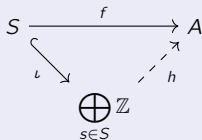
Definition

The **free abelian group** on $S \neq \emptyset$ is $F = \bigoplus_{s \in S} \mathbb{Z}$.

Theorem

Let $S \neq \emptyset$. The group $\bigoplus_{s \in S} \mathbb{Z}$ with $\iota(s) = \mathbf{e}_s$ is a free object for S in **Ab**.

That is, given any $f: S \rightarrow A$ there exists a unique $h: \bigoplus_s \mathbb{Z} \rightarrow A$ such that $f = h \circ \iota$.



Proof (sketch)

Existence and uniqueness of the desired function h is constructive:

$$h\left(\sum_{i=1}^n a_{s_i} \mathbf{e}_{s_i}\right) = \left(\sum_{i=1}^n a_{s_i} h(\mathbf{e}_{s_i})\right) = \left(\sum_{i=1}^n a_{s_i} h(\iota(s_i))\right) = \sum_{i=1}^n a_{s_i} f(s_i). \quad \square$$

Group presentations, formalized

Definition

For any subset $R \subseteq F_S$, the group $G = \langle S \mid R \rangle$ is the quotient F_S/N where

$$N := \bigcap_{R \leq N_\alpha \trianglelefteq F_S} N_\alpha.$$

Elements in R are called **relators**.

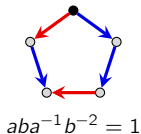
Big idea

The group $\langle S \mid R \rangle$ is the quotient of F_S by the *smallest normal subgroup containing R* .

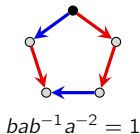
Exercise: show that

$$G = \langle a, b \mid ab = b^2a, ba = a^2b \rangle = \langle 1 \rangle.$$

In terms of Cayley graphs and motifs, this means that



and



\implies



$$G = \langle a, b \mid a = b = 1 \rangle = \langle 1 \rangle$$

Group presentations, formalized

Given $G_1 = \langle S \mid R_1 \rangle$, define $G_2 = \langle S \mid R_2 \rangle$ by adding relations: $R_1 \subseteq R_2$.

We have two quotient maps,

$$\pi_1: F_S \longrightarrow F_S/N_1 \cong G_1, \quad \pi_2: F_S \longrightarrow F_S/N_2 \cong G_2,$$

Since $N_1 = \text{Ker}(\pi_1) \subseteq \text{Ker}(\pi_2) = N_2$, the co-universal property of quotients gives us:

A commutative diagram with F_S at the top left, G_1 at the bottom left, and G_2 at the top right. A solid arrow labeled π_1 points from F_S to G_1 . A solid arrow labeled π_2 points from F_S to G_2 . A dashed arrow labeled h points from G_1 to G_2 . The diagram is completed by a solid arrow from F_S to G_2 labeled π_2 .

Now, suppose $G_1 = \langle S_1 \mid R \rangle$ and $G_2 = \langle S_2 \mid R \rangle$ with $S_1 \supseteq S_2$.

Defining $R' = S_1 \setminus S_2$, we have

$$G_1 = \langle S_1 \mid R \rangle, \quad G_2 = \langle S_1 \mid R \cup R' \rangle,$$

and hence a quotient $G_1 \twoheadrightarrow G_2$.

Proposition

Given $G_1 = \langle S_1 \mid R_1 \rangle$ and $G_2 = \langle S_2 \mid R_2 \rangle$ for which $S_1 \supseteq S_2$ and $R_1 \subseteq R_2$, there is a quotient $G_1 \twoheadrightarrow G_2$. □

Group presentations, formalized

In many cases, two generating sets that we wish to compare are not subsets of each other.

For example, if $S_1 = \{a, b, c\}$ and $S_2 = \{r, f\}$, then $S_1 \not\supseteq S_2$.

However, there is $\theta: S_1 \rightarrow S_2$ that can be thought of as a “relabeling.”

Saying that “every relation in G_1 is a relation in G_2 ” means that every $\theta(r_1)$ is a relator.

We say that such a map θ respects relations, because it extends to a map $\theta: R_1 \rightarrow R_2$.

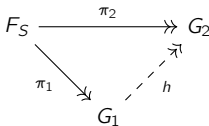
Proposition

Suppose $G_1 = \langle S_1 \mid R_1 \rangle$ and $G_2 = \langle S_2 \mid R_2 \rangle$ and the following holds:

1. there exists $\theta: S_1 \rightarrow S_2$ extending to $\theta: R_1 \rightarrow R_2$,
2. $r_2 := \theta(r_1) = 1$ for all $r \in R_1$.

Then there is a quotient $h: G_1 \rightarrow G_2$.

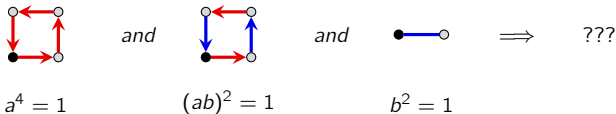
□



Group presentations, formalized

Consider the “mystery group” $M = \langle a, b \mid a^4 = b^2 = 1, (ab)^2 = 1 \rangle$.

Visually, we are asking what the largest Cayley graph is given several motifs:



Elements in M can be written as $a^i b^j$, for $i = \{0, 1, 2, 3\}$ and $j = \{0, 1\}$. Thus, $|M| \leq 8$.

We'll show $|M|$ is a multiple of 8, by constructing a homomorphism

$$\theta: M \longrightarrow \left\langle \left[\begin{array}{cc} i & 0 \\ 0 & -i \end{array} \right], \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right] \right\rangle \cong D_4, \quad \theta(a) = \left[\begin{array}{cc} i & 0 \\ 0 & -i \end{array} \right], \quad \theta(b) = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right].$$

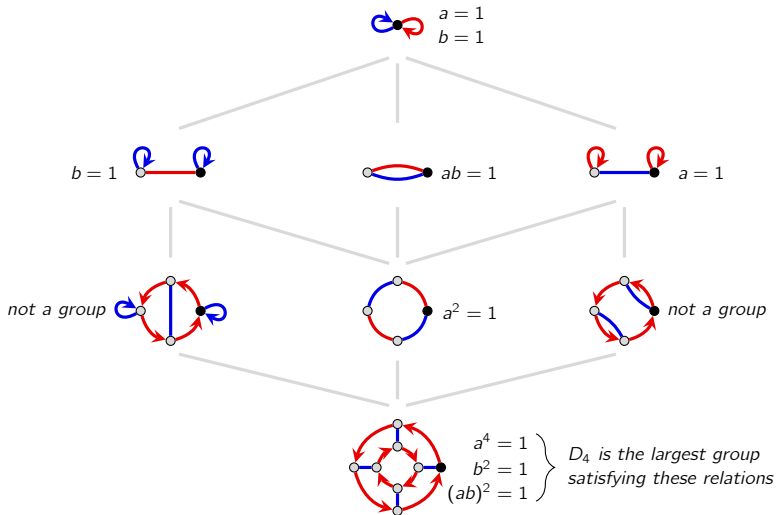
This respects relations because

$$(\theta(a))^4 = (\theta(b))^2 = (\theta(a)\theta(b))^2 = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right].$$

Thus, there is a quotient $g: M \rightarrow D_4$, and so $M \cong D_4$.

Group presentations, formalized

Every group $G = \langle a, b \rangle$ satisfying $a^4 = 1$, $b^2 = 1$, and $(ab)^2 = 1$ is a quotient of D_4 .



Group presentations, formalized

Overview of the strategy

Given a “mystery” $M = \langle S_1 \mid R_1 \rangle$ that we suspect is a “familiar” $F = \langle S_2 \mid R_2 \rangle$:

1. Using the relations, show that $|M| \leq |F|$.
2. Identify generators of F that satisfy the relations in M , via a “relabeling map” $\theta: S_1 \rightarrow S_2$ that extends to $\theta: R_1 \rightarrow R_2$.

Together, $|M| \leq |F|$ and $M \rightarrow F$ forces $M \cong F$.

Consider the group $M = \langle a, b, c \mid a^4 = c^2 = 1, a^2 = b^2, ab = ba, ac = ca, a^2b = abc \rangle$.



$$a^4 = 1$$



$$c^2 = 1$$



$$a^2 = b^2$$



$$ab = ba$$



$$ac = ca$$



$$a^2b = abc$$

Homework: Establish $|M| \leq 16$ by showing that every word in M can be written

$$a^i b^j c^k, \quad i \in \{0, 1, 2, 3\}, \quad j \in \{0, 1\}, \quad k \in \{0, 1\},$$

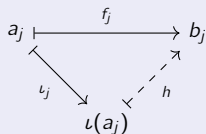
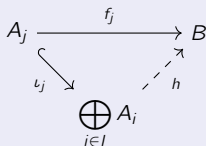
Then, find a “familiar group” F of order 16 whose generator satisfies these relations.

That will define a quotient $\pi: M \rightarrow F$, and hence $|M| \geq |F| = 16$.

Free products

Proposition

The coproduct of $\{A_i \mid i \in I\}$ in \mathbf{Ab} is the direct sum, $S = \bigoplus_i A_i$:



Proof

Let C be the coproduct of the factors, with $\iota_j: A_j \hookrightarrow C$.

Consider the group $B \leq C$ generated by the images of all individual factors,

$$B = \langle \iota_j(A_j) \mid j \in I \rangle, \quad \text{and let } g: B \hookrightarrow C.$$

Each $b \in B$ can be written as

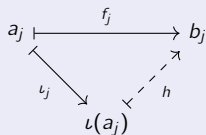
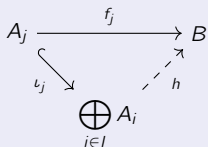
$$b = \sum_{j=1}^k \iota(a_j), \quad a_j \in A_{i_j},$$

and so $B \cong S$. Let $f_j: A_j \hookrightarrow B$ be the natural inclusion map.

Free products

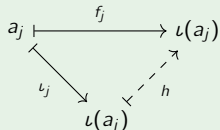
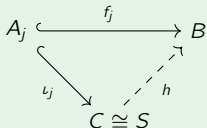
Proposition

The coproduct of $\{A_i \mid i \in I\}$ in \mathbf{Ab} is the direct sum, $S = \bigoplus_{i \in I} A_i$:



Proof (cont.)

By the co-universal property of coproducts, we have:

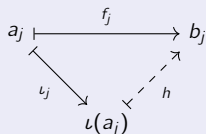
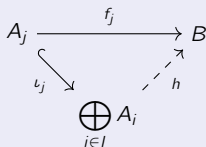


It is clear that $h \circ g = \text{Id}_B$. It suffices to show that $g \circ h = \text{Id}_C$.

Free products

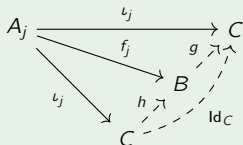
Proposition

The coproduct of $\{A_i \mid i \in I\}$ in \mathbf{Ab} is the direct sum, $S = \bigoplus_i A_i$:



Proof (cont.)

Since $\iota_j = g \circ f_j$ and $f_j = h \circ \iota_j$, the “small triangles” in the following diagram commute:



It follows that $\iota_j = g \circ h \circ \iota_j$, but we also have $\iota_j = \text{Id}_C \circ \iota_j$.

By uniqueness from the co-universal property, $g \circ h = \text{Id}_C$. □

Free products

The coproduct of two groups A and B in \mathbf{Grp} is a construction called the **free product**.

Given groups $A = \langle S_1 \mid R_1 \rangle$ and $B = \langle S_2 \mid R_2 \rangle$, their *free product* is

$$A * B := \langle S_1 \sqcup S_2 \mid R_1 \sqcup R_2 \rangle.$$

If $A = \langle a \mid \rangle = C_\infty \cong \mathbb{Z}$ and $B = \langle b \mid \rangle \cong C_\infty$, then $A * B$ is the free group $F_2 = \langle a, b \mid \rangle$.

If A and B are nontrivial, their free product is infinite, because

$$a, ab, aba, abab, ababa, ababab, \dots$$

are all distinct, assuming $a, b \neq 1$.

The free product of the groups $A = \langle a \mid a^2 = 1 \rangle \cong C_2$ and $B = \langle b \mid b^2 = 1 \rangle \cong C_2$ is

$$A * B = \langle a, b \mid a^2 = 1, b^2 = 1 \rangle \cong D_\infty$$



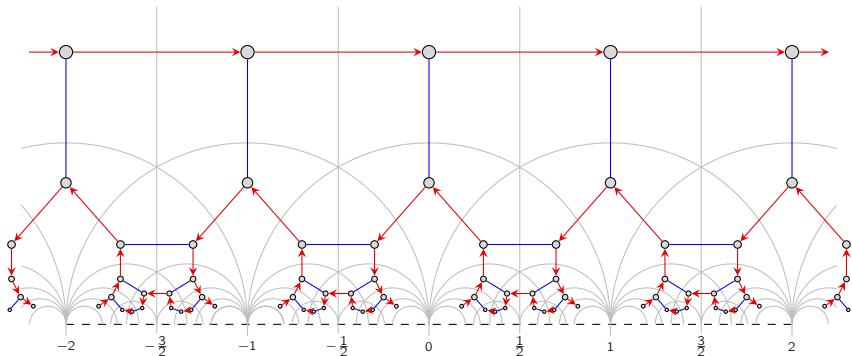
Free products

The free product $C_3 * C_2$ is isomorphic to the projective linear group

$$\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z}) / \langle -I \rangle, \quad \text{where } \mathrm{SL}_2(\mathbb{Z}) = \left\langle \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_S, \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_T \right\rangle.$$

This is in no way obvious from the generators that we've seen, which represent

$$S: z \mapsto \frac{0z - 1}{z + 0} = -\frac{1}{z}, \quad \text{and} \quad T: z \mapsto \frac{z + 1}{0z + 1} = z + 1.$$



Free products

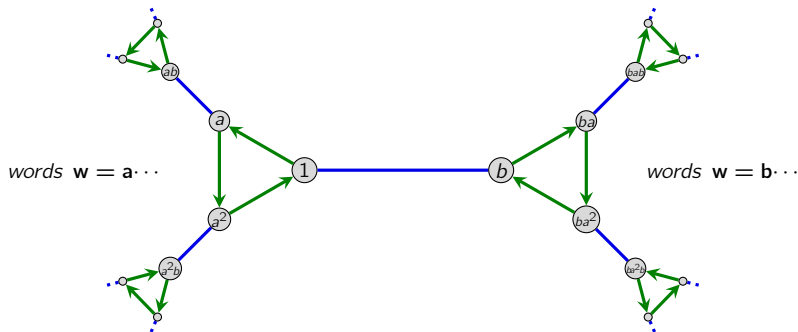
Let's see why the free product $C_3 * C_2$ is isomorphic to the projective linear group

$$\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z}) / \langle -I \rangle.$$

Elements of $\mathrm{PSL}_2(\mathbb{Z})$ are cosets of $\langle -I \rangle = \pm I$. Let

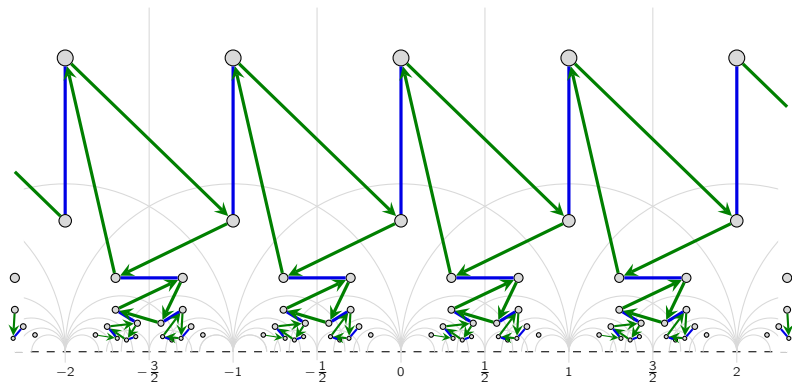
$$\mathrm{SL}_2(\mathbb{Z}) = \langle S, T \mid S^2 = (ST)^6 = I \rangle, \quad S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad ST = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix},$$

Then $\mathrm{PSL}_2(\mathbb{Z}) \cong \langle A, B \rangle$, where $A = \pm ST$ and $B = \pm S$.



Free products

A Cayley graph of $\mathrm{PSL}_2(\mathbb{Z}) = \langle A, B \mid A^3 = B^2 = 1 \rangle \cong C_3 * C_2$:



To verify $\mathrm{PSL}_2(\mathbb{Z}) \cong C_3 * C_2$, it suffices to show that we can't nontrivially write

$$I = A^{i_1} B A^{i_2} B \cdots A^{i_{m-1}} B A^{i_m}, \quad i_1, i_m \in \{0, 1, 2\}, \quad i_k \in \{1, 2\}.$$

This will be left as HW.

Free products

Definition

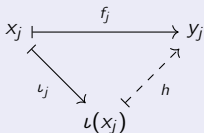
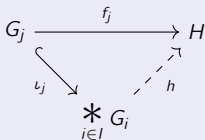
The **free product** of a family $G_i = \langle S_i \mid R_i \rangle$ of groups is

$$\ast_{i \in I} G_i = \left\langle \bigsqcup_{i \in I} S_i \mid \bigsqcup_{i \in I} R_i \right\rangle, \quad \text{where } \iota_j: G_j \hookrightarrow \ast_{i \in I} G_i, \quad \iota_j(x_j) = x_j,$$

Exercise

The coproduct of $\{G_i \mid i \in I\}$ in **Grp** is their free product.

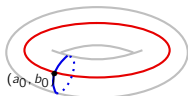
That is, given any H and $\{f_j: G_j \rightarrow H \mid j \in I\}$, there is a unique $h: \ast_i G_i \rightarrow H$ such that $f_j = \iota_j \circ h$ for all $j \in I$.



Fiber coproducts in **Grp**: free products with amalgamation

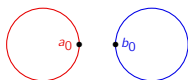
Suppose A and B are disjoint circles. Gluing them at a point is called their **wedge sum**.

Product



$A \times B$: “Cartesian product”

Coproduct



$A \sqcup B$: “disjoint union”

Coproduct w/ amalgamation



$A \vee B$: “wedge sum”

In general, we can identify or “glue” two objects along a common subset. Gluing two disks along their boundaries gives a sphere.

Suppose $A \trianglelefteq G_i$ for $i = 1, 2$, with embeddings $\alpha_j: A \hookrightarrow G_j$.

Goal: Take the the coproduct of G_1 with G_2 , and “identify” the common subgroup A .

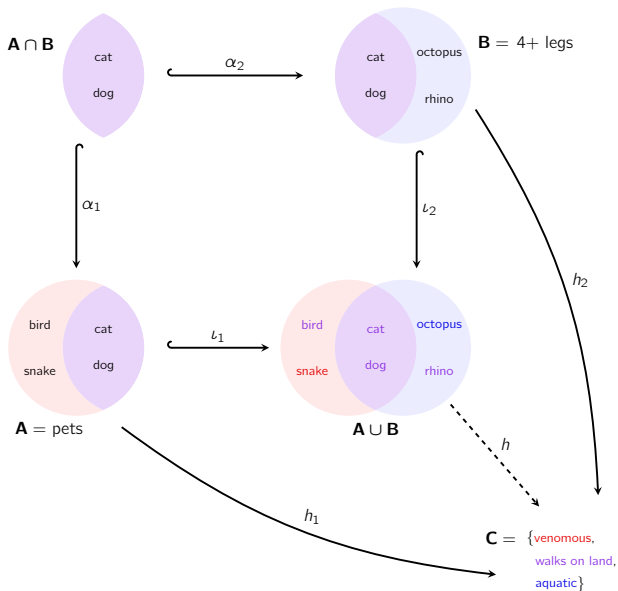
We can “force” $\alpha_1(a) \in G_1$ and $\alpha_2(a) \in G_2$ (in $G_1 * G_2$) to be the same by adding relations

$$\alpha_1(a)\alpha_2(a)^{-1} = 1, \quad \text{for all } a \in A,$$

and then quotient $A * B$ by the smallest normal subgroup N that contains these relators.

The group $G_1 *_A G_2 := (G_1 * G_2)/N$ is the **free product of G_1 and G_2 amalgamated at A** .

Fiber coproducts in **Set**: unions



Fiber coproducts in **Grp**: free products with amalgamation

$G_1 * G_2$ is the smallest group in which both G_1 and G_2 embeds into “independently.”

I.e., for any other H with this property, those embeddings factor through via $G_1 * G_2 \rightarrow H$.

For $i = 1, 2$, let $\iota_i: G_i \rightarrow (G_1 * G_2)/N = G_1 *_A G_2$ be the map $\iota_i: g_i \mapsto g_i N$.

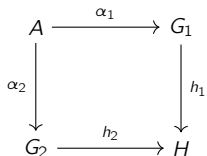
$$\begin{array}{ccc} A & \xrightarrow{\alpha_1} & G_1 \\ \alpha_2 \downarrow & & \downarrow \iota_1 \\ G_2 & \xrightarrow{\iota_2} & (G_1 * G_2)/N \end{array} \quad (2)$$

$G_1 *_A G_2$ is the smallest group in which both G_1 and G_2 embeds into “independently,” while keeping A identified.

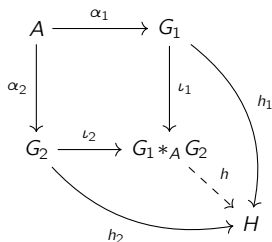
The **central product**, e.g., $DQ_8 \cong D_4 \circ C_4 \cong Q_8 \circ C_4$, is a *direct product with amalgamation*.

Fiber coproducts in **Grp**: free products with amalgamation

Suppose G_1 and G_2 embed into H while keeping A identified:



Then $\exists! h: G_1 *_A G_2 \rightarrow H$ that makes the following diagram commute:



Fiber coproducts in a general category

Definition

Let $A, B_1, B_2 \in \text{Ob}(\mathcal{C})$ and $\alpha_i \in \text{Hom}_{\mathcal{C}}(A, B_i)$ for $i = 1, 2$. A **fiber coproduct** (or **pushout**) for them is a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha_1} & B_1 \\ \alpha_2 \downarrow & & \downarrow \iota_1 \\ B_2 & \xrightarrow{\iota_2} & C \end{array}$$

satisfying the following couniversal property:

For any $D \in \text{Ob}(\mathcal{C})$ and $h_i \in \text{Hom}_{\mathcal{C}}(B_i, D)$ such that if $h_1 \circ \alpha_1 = h_2 \circ \alpha_2$, there exists a unique $h \in \text{Hom}_{\mathcal{C}}(C, D)$ such that $h \circ \iota_i = h_i$.

$$\begin{array}{ccc} A & \xrightarrow{\alpha_1} & B_1 \\ \alpha_2 \downarrow & & \downarrow \iota_1 \\ B_2 & \xrightarrow{\iota_2} & C \end{array} \begin{array}{l} \xrightarrow{h_1} \\ \xrightarrow{h_2} \\ \xrightarrow{h} \end{array} D$$

Fiber coproducts (pushouts)

Proposition

Pushouts are unique up to equivalence.

Proof

Suppose we have two pushouts for A, B_1, B_2 :

$$\begin{array}{ccc} A & \xrightarrow{\alpha_1} & B_1 \\ \alpha_2 \downarrow & & \downarrow \iota_1 \\ B_2 & \xrightarrow{\iota_2} & C \end{array}$$

$$\begin{array}{ccc} A & \xrightarrow{\alpha_1} & B_1 \\ \alpha_2 \downarrow & & \downarrow \delta_1 \\ B_2 & \xrightarrow{\delta_2} & D \end{array}$$

By the co-universal property, we have $h \in \text{Hom}_C(C, D)$ and $g \in \text{Hom}_C(D, C)$ such that:

$$\begin{array}{ccc} A & \xrightarrow{\alpha_1} & B_1 \\ \alpha_2 \downarrow & & \downarrow \iota_1 \\ B_2 & \xrightarrow{\iota_2} & C \end{array} \quad \begin{array}{c} \delta_1 \\ \downarrow \\ \delta_2 \end{array}$$

Diagram illustrating the relationship between the two pushouts. The left pushout is shown with $A \xrightarrow{\alpha_1} B_1$, $\alpha_2 \downarrow$, $B_2 \xrightarrow{\iota_2} C$, and $\downarrow \iota_1$. The right pushout is shown with $A \xrightarrow{\alpha_1} B_1$, $\alpha_2 \downarrow$, $B_2 \xrightarrow{\delta_2} D$, and $\downarrow \delta_1$. A curved arrow labeled δ_1 points from B_1 to D . A curved arrow labeled δ_2 points from B_2 to D . A dashed arrow labeled h points from C to D .

$$\begin{array}{ccc} A & \xrightarrow{\alpha_1} & B_1 \\ \alpha_2 \downarrow & & \downarrow \delta_1 \\ B_2 & \xrightarrow{\delta_2} & D \end{array} \quad \begin{array}{c} \iota_1 \\ \downarrow \\ \iota_2 \end{array}$$

Diagram illustrating the relationship between the two pushouts. The left pushout is shown with $A \xrightarrow{\alpha_1} B_1$, $\alpha_2 \downarrow$, $B_2 \xrightarrow{\delta_2} D$, and $\downarrow \delta_1$. The right pushout is shown with $A \xrightarrow{\alpha_1} B_1$, $\alpha_2 \downarrow$, $B_2 \xrightarrow{\iota_2} C$, and $\downarrow \iota_1$. A curved arrow labeled ι_1 points from B_1 to C . A curved arrow labeled ι_2 points from B_2 to C . A dashed arrow labeled g points from D to C .

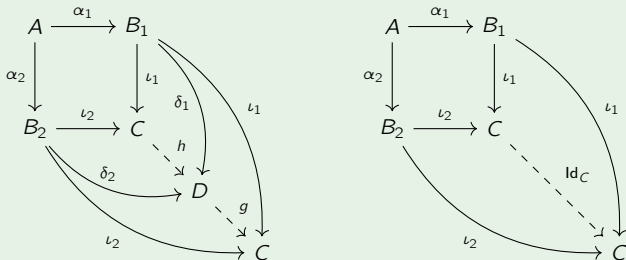
Fiber coproducts (pushouts)

Proposition

Pushouts are unique up to equivalence.

Proof (cont.)

We can “stack” these diagrams to get:



By uniqueness from the co-universal property, $g \circ h = \text{Id}_C$.

Stacking them the other way gives $h \circ g = \text{Id}_D$.

Therefore, h and g are inverse isomorphisms, and hence $C \cong D$. □

Fiber coproducts (pushouts)

In **Set** and **Top**, pushouts are ordinary unions:

$$\begin{array}{ccc} Y \cap Z & \xrightarrow{\alpha_Z} & Z \\ \alpha_Y \downarrow & & \downarrow \iota_Z \\ Y & \xrightarrow{\iota_Y} & Y \cup Z \end{array}$$

Siefert van-Kampen theorem

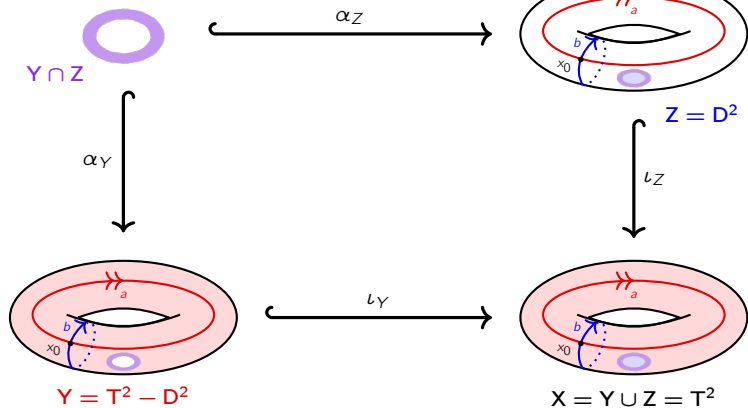
The functor $\pi_1: \mathbf{Top} \rightarrow \mathbf{Grp}$ preserves pushouts.

$$\begin{array}{ccc} \mathcal{F}(Y \cap Z) & \xrightarrow{\alpha_Z^*} & \mathcal{F}(Z) \\ \alpha_Y^* \downarrow & & \downarrow \iota_Z^* \\ \mathcal{F}(Y) & \xrightarrow{\iota_Y^*} & \mathcal{F}(Y \cup Z) \end{array}$$

$$\begin{array}{ccc} \pi_1(Y \cap Z) & \xrightarrow{\alpha_Z^*} & \pi_1(Z) \\ \alpha_Y^* \downarrow & & \downarrow \iota_Z^* \\ \pi_1(Y) & \xrightarrow{\iota_Y^*} & \pi_1(Y \cup Z) \end{array}$$

The Siefert van-Kampen theorem

$$\pi_1(Y \cap Z) = \langle aba^{-1}b^{-1} \rangle \cong \mathbb{Z}$$



$$\pi_1(Y) = \langle a, b \mid \rangle$$

$$\begin{aligned} \pi_1(Y \cup Z) &= \pi_1(Y) *_{\pi_1(Y \cap Z)} \pi_1(Z) \cong \langle a, b \mid \rangle *_{\langle aba^{-1}b^{-1} \rangle} \langle 1 \rangle \\ &= \langle a, b \mid aba^{-1}b^{-1} = 1 \rangle \cong \mathbb{Z} \times \mathbb{Z} \end{aligned}$$

Fiber products / pullbacks (HW)

Definition

Let $A_1, A_2, B \in \text{Ob}(\mathcal{C})$ and $\alpha_i \in \text{Hom}_{\mathcal{C}}(A_i, B)$ for $i = 1, 2$. A **fiber product** (or **pullback**) for them is a commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{\pi_1} & A_1 \\ \pi_2 \downarrow & & \downarrow \alpha_1 \\ A_2 & \xrightarrow{\alpha_2} & B \end{array}$$

satisfying the following universal property:

For any $Q \in \text{Ob}(\mathcal{C})$ and $h_i \in \text{Hom}_{\mathcal{C}}(Q, A_i)$ such that if $\alpha_1 \circ h_1 = \alpha_2 \circ h_2$, there exists a unique $h \in \text{Hom}_{\mathcal{C}}(Q, P)$ such that $h_i = \pi_i \circ h$.

$$\begin{array}{ccc} & & h_1 \\ & \searrow & \downarrow \\ Q & \xrightarrow{h} & P \\ & \swarrow & \downarrow \pi_1 \\ & & A_1 \\ & & \downarrow \alpha_1 \\ & & B \\ & \swarrow & \downarrow \alpha_2 \\ & & A_2 \\ & \searrow & \downarrow \pi_2 \\ & & P \end{array}$$