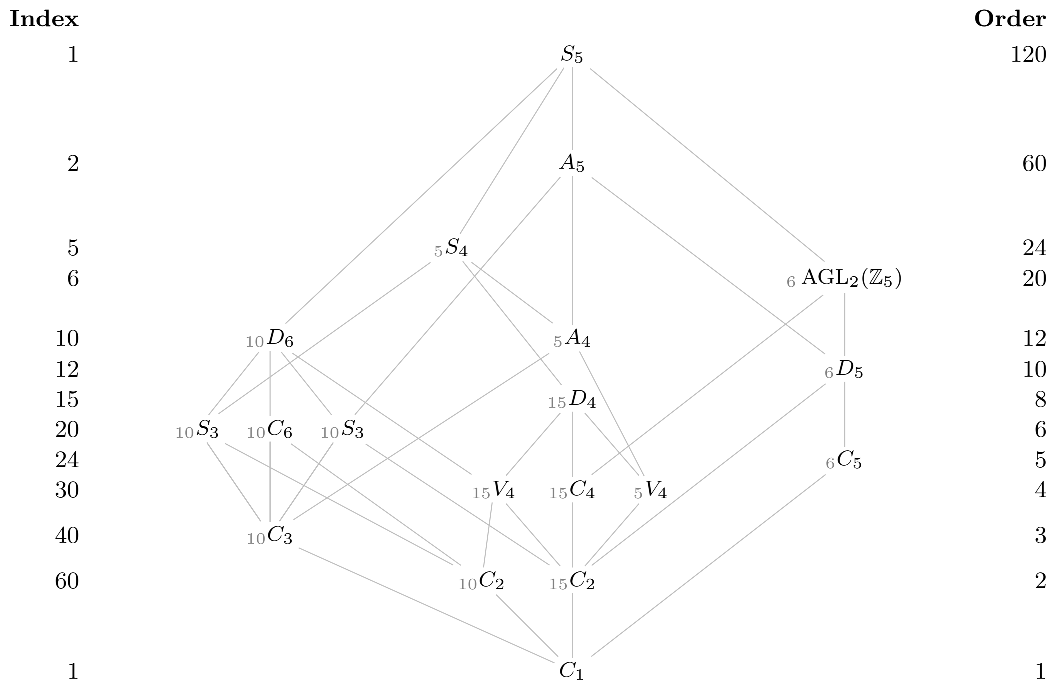
Math 4120, Final Exam. April 28, 2025

Write your answers for these problems *directly on this paper*. You should be able to fit them in the space given.

1. (28 pts) Answer the following questions about the symmetric group S_5, whose subgroup diagram is below.



(a) There are __________ Sylow 2-subgroup(s), which are isomorphic to __________.

(b) There are __________ Sylow 3-subgroup(s), which are isomorphic to __________.

(c) There are __________ Sylow 5-subgroup(s), which are isomorphic to __________.

(d) There are _____ subgroups of S_5 that fall into _____ conjugacy classes (don't forget S_5 and C_1).

(e) S_5 has exactly ____________________ normal subgroups.

(f) The action of S_5 on its subgroups by conjugation has _____ orbit(s) and _____ fixed points(s).

(g) The largest order of any element in S_5 is ____________________.

(h) S_5 has __________ subgroups of order 3, and __________ elements of order 3.

(i) The subgroup $\langle(12)\rangle \cong$__________, and its normalizer is isomorphic to __________.

(j) The subgroup $\langle(12)(34)\rangle \cong$__________, and its normalizer is isomorphic to __________.

(k) The subgroup $\langle(123),(45)\rangle$ *(is)(is not)* [$\longleftarrow$ *circle one*] abelian, and is isomorphic to ________.

(l) The center of S_5 is $Z(S_5) \cong$____________________.

(m) The subgroup $\langle(12),(34)\rangle \cong$_____, and $\langle(12)\rangle,(23),(34)\rangle \cong$ _____, and $\langle(123),(12)\rangle \cong$_____.

(n) A subgroup isomorphic to D_5 is $\Big\langle \qquad\qquad , \qquad\qquad \Big\rangle$. (*Hint*: draw a pentagon.)

(o) List all way(s) S_5 decomposes into a semidirect product of two proper subgroups: __________.

(p) The commutator subgroup of $G = S_5$ is $G' =$______, and the abelianization is $G/G' \cong$______.

(q) An example of a proper subgroup that is equal to its own normalizer is _______________.

2. (9 pts) This class has been all about *groups*. Loosely speaking, a group is a set G with an associative binary operation that satisfies three additional properties. List each of these below, and then give a *formal mathematical* definition of each. For full credit, you must properly use terms of $\forall$ and $\exists$, where appropriate. Your definitions should read as if they were in a textbook.

   (i)

   (ii)

   (iii)

3. (12 pts) Let H be a subgroup of G, and $x \in G$.

   (a) Finish the following definition: $xH = \{ \qquad\qquad\qquad\qquad\qquad\qquad \}$.

   (b) Show that if $b \in aH$, then $aH = bH$. [*Hint*: Show both $\subseteq$ and $\supseteq$, separately.]

4. (10 pts) Make a complete list of all abelian groups of order $72 = 2^3 \cdot 3^2$, up to isomorphism. Every group should appear exactly once on your list. Then circle each group that is cyclic.

5. (12 pts) Recall that there are seven distinct frieze groups, but only four up to isomorphism. You are *not* expected to have remembered or memorized these, or even have studied them. However, you should now be familiar enough with algebraic concepts to work these out, and be able to check your work and verify whether your initial answers are correct, using problem solving skills.

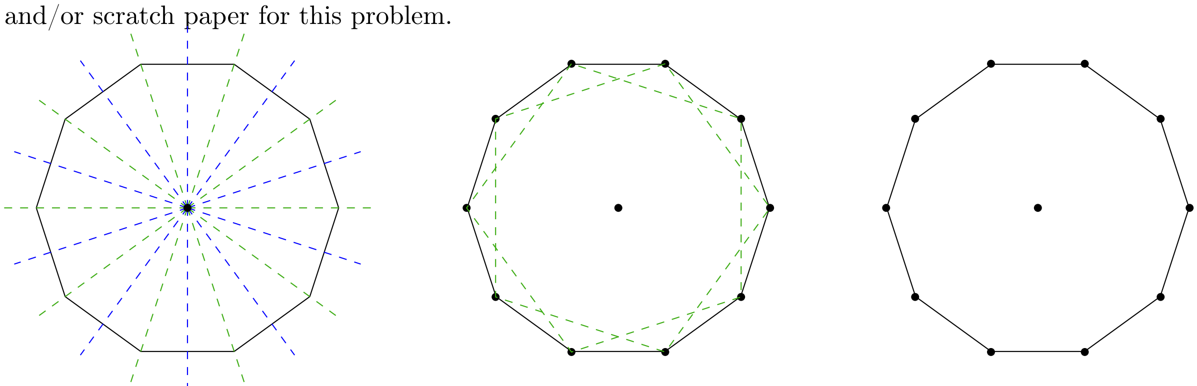
   (a) Draw a frieze whose frieze group *cannot* be generated by just two symmetries. Write a *minimal* generating set for this group.

   (b) Draw a frieze whose frieze group has two generators but is *nonabelian*. Sketch a Cayley graph and write a presentation for this group. Write it as a semidirect product of two familiar groups.

   (c) Draw a frieze whose frieze group is *abelian* but *not cyclic*. Sketch a Cayley graph and write a presentation for this group. Write it as a direct product of two familiar groups.

   (d) Draw a frieze whose frieze group is *cyclic*. Sketch a Cayley graph and write a presentation for this group.

6. (28 pts) Throughout, let $G = D_{10} = \langle r, f \rangle$, the set of symmetries of a regular 10-gon, where r is a counterclockwise $36°$ rotation, and f is a reflection across the vertical (blue) axis, in the picture below (left). Also included below are two other 10-gons, in case you want to use them as a visual reference and/or scratch paper for this problem.



Tip: For many of these parts, it is very helpful to think about the group *geometrically*—in terms of the actual symmetries.

(a) The $180°$ rotation *(is)(is not)* [$\longleftarrow$ *circle one*] an element of D_{10}.

(b) The subgroup $\langle f, r^2 f \rangle$, generated by two reflections, is isomorphic to ____________________.

(c) The group D_{10} has ________ element(s) of order 2, and *(all)(some)(none)* of them are reflections.

(d) The group D_{10} has ____________________________ subgroup(s) isomorphic to C_2.

(e) The group D_{10} has __________________________________ element(s) of order 4.

(f) The group D_{10} has ____________________________ subgroup(s) isomorphic to C_{10}.

(g) The group D_{10} has ________ element(s) of order 10: ________________ [$\longleftarrow$ *list them all*]

(h) The group D_{10} has ____________________________ subgroup(s) isomorphic to C_5.

(i) The group D_{10} has ____________ element(s) of order 5: ________________________.

(j) The group D_{10} has ____________________________ subgroup(s) isomorphic to D_5.

(k) The group D_{10} has ____________________________ subgroup(s) isomorphic to V_4.

(l) A *group presentation* for D_{10} is $\left\langle r, f \mid \qquad\qquad\qquad\qquad \right\rangle$.

(m) Draw a *cycle graph* for D_{10}. You do not have to label the nodes (but you're welcome to!), and edges should be undirected.

(n) Draw a *Cayley graph* for $D_{10} = \langle r, f \rangle$. You don't have to label the nodes with elements, but you may find that doing so will help you later in this problem, or in checking earlier parts.

(o) In a single short sentence, describe what a Cayley graph for $D_{10} = \langle s, t \rangle$ would look like, where $s = f$ and $t = rf$ are "adjacent reflections." You do *not* have to actually draw it!

(p) Draw the subgroup lattice for $D_{10} = \langle r, f \rangle$ below. Write the subgroups by generator(s), not by isomorphism type. Your answers to earlier parts of the problem should be helpful. [*Hint*: Organizationally, it is a little easier to write down the groups of order 10 *before* those of order 2.] Circle the conjugacy classes of the non-normal subgroups.

Index = 1	$D_{10} = \langle r, f \rangle$	**Order** = 20
2		10
4		5
5		4
10		2
20	$\langle 1 \rangle$	1

7. (25 pts) Answer the following.

(a) If $H \leq G$, then $xH = H$ if and only if ______________________________.

(b) By definition, the index $[G : H]$ of H in G is the number of ____________________.

(c) By definition, then *order* of a group G is the __________________________.

(d) There are ________ groups of order 4, and they are ________________ [$\longleftarrow$ *list them*].

(e) The five groups of order 8 are: ______________________________.

(f) If G/H is non-cyclic, then the smallest that $[G : H]$ can be is ____________________.

(g) For any $n \geq 2$, the index of the alternating group $A_n \leq S_n$ is $[S_n : A_n] =$______________.

(h) A group G is *abelian* iff the center $Z(G)$ is ____________________________.

(i) A subgroup $H \leq G$ is *normal* iff its normalizer $N_G(H)$ is ____________________.

(j) An element $g \in G$ is *central* iff its centralizer $C_G(g)$ is ________________, or equivalently, if its commutator subgroup G' is ______________________________.

(k) If $K \trianglelefteq H \trianglelefteq G$, then K *(is)(is not)(need not be)* [$\longleftarrow$ *circle one*] normal in G.

(l) Recall that the automorphism group of the abelian group V_4 is $\mathrm{Aut}(V_4) \cong S_3$. Knowing this, we can deduce that the *inner automorphism* group is $\mathrm{Inn}(V_4) \cong$____________________,
and the *outer automorphism group* is $\mathrm{Out}(V_4) := \mathrm{Aut}(V_4)/\,\mathrm{Inn}(V_4) \cong$______________.

(m) If G acts on S, then for any $s \in S$, the stabilizer subgroup $\mathrm{stab}(s)$ *(is)(is not)(need not be)* normal.

(n) If $xy^{-1} \in H$ for every $x, y \in H$, then H is ____________________________.

(o) A group action is *transitive* if and only if its action graph is ____________________.

(p) Every transitive G-action is isomorphic to G acting on ______________________.

(q) The second Sylow theorem says that all ______________________ are conjugate.

(r) If I is an ideal of R, then I *(is)(is not)(need not be)* [$\longleftarrow$ *circle one*] a subring of R.

(s) If I is an ideal of R, then I *(is)(is not)(need not be)* a subgroup of $(R, +)$.

(t) If I is a left ideal of R, then the quotient R/I *(is)(is not)(need not be)* a ring.

(u) An ideal I of R is *maximal* if and only if R/I is __________________________.

(v) The smallest finite field that is *not* of the form $\mathbb{Z}_p = \{0, \dots, p-1\}$ for some $p \in \mathbb{N}$ has order ____.

8. (16 pts) In this problem, you will prove the fundamental homomorphism theorem (FHT) for rings. Throughout, assume that R is a ring with $1 \neq 0$, and $\phi\colon R \to S$ is a ring homomorphism.

   (a) The *kernel* of ϕ is the set $\mathrm{Ker}(\phi) = \Big\{ \qquad\qquad\qquad\qquad\qquad\qquad \Big\}$.

   (b) We know that $\mathrm{Ker}(\phi)$ is a subgroup of $(R,+)$. Prove that it is a left ideal of R. (It is actually a two-sided ideal, but the proof that it is a right ideal is completely analogous, so you may skip that.)

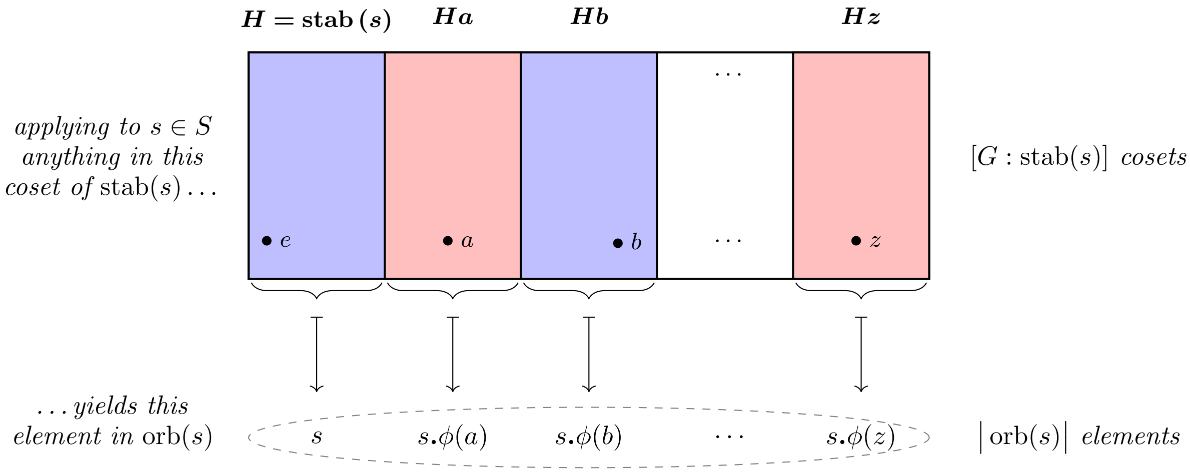
   (c) The proof of the FHT for groups involves defining a map $\iota\colon R/I \longrightarrow \mathrm{Im}(\phi)$ and proving it is a bijection. How should this map ι be defined? [Be careful about the difference between rI vs. $r+I$!]

   (d) You may assume the FHT for groups, i.e., that the map $\iota\colon R/I \longrightarrow \mathrm{Im}(\phi)$ defined above is a well-defined bijection, and thus is a group isomorphism. Prove that it is additionally a *ring isomorphism*.

9. (14 pts) Let G act on a set S, and let $H = \mathrm{stab}(s)$ for some $s \in S$. Prove the *orbit-stabilizer theorem*, which says that

$$[G : \mathrm{stab}(s)] = |\,\mathrm{orb}(s)|.$$

[*Hint*: Start by defining a map $f\colon H\backslash G \to \mathrm{orb}(s)$, and then prove that it has the desired properties. You might find the following visual useful, which outlines how the map should be defined.]



10. (16 pts) Write formal mathematical definitions for the following concepts.

    (a) The *center* of a subgroup G is $Z(G) = \Big\{ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Big\}$.

    (b) The *normalizer* of a subgroup $H \leq G$ is $N_G(H) = \Big\{ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Big\}$.

    (c) The *conjugacy class* of an element $x \in G$ is $\mathrm{cl}_G(x) = \Big\{ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Big\}$.

    (d) A *homomorphism* from a group G to another group H is ...

    (e) An *action* of a group G on a set S is ...

    (f) If G acts on S, then the *orbit* of $s \in S$ is $\mathrm{orb}(s) = \Big\{ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Big\}$.

    (g) If G acts on S, then the *stabilizer* of $s \in S$ is $\mathrm{stab}(s) = \Big\{ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Big\}$.

    (h) A *left ideal* I of a ring R is ...

11. (14 pts) Let $R = \mathbb{Z}$ be the ring of integers, with ideals $I = (4) = 4\mathbb{Z}$ and $J = (6) = 6\mathbb{Z}$. For each of the sets below, write enough elements to make it clear that you know what the set is. For example,

$$I = \big\{\dots, -8, -4, 0, 4, 8, \dots\big\}, \qquad \text{and} \qquad J = \big\{\dots, -12, -6, 0, 6, 12, \dots\big\}.$$

Then, besides each one, determine if it is a subring of R, and if it is an ideal. (Answer Y or N.)

	Subring? (Y/N)	**Ideal?** (Y/N)
(a) The set $3 + I = \big\{ \qquad\qquad\qquad\qquad \big\}$.		
(b) The set $(3 + I)(3 + I) = \big\{ \qquad\qquad\qquad\qquad \big\}$.		
(c) The set $3I = \big\{ \qquad\qquad\qquad\qquad \big\}$.		
(d) The set $(3 + I)(4 + I) = \big\{ \qquad\qquad\qquad\qquad \big\}$.		
(e) The set $I + J = \big\{ \qquad\qquad\qquad\qquad \big\}$.		
(f) The set $IJ = \big\{ \qquad\qquad\qquad\qquad \big\}$.		
(g) The set $I \cap J = \big\{ \qquad\qquad\qquad\qquad \big\}$.		

12. (12 pts) The nonzero complex numbers $\mathbb{C}^*$ are a group under multiplication. Let $U = \{e^{i\theta} \mid \theta \in \mathbb{R}\}$, which are the complex numbers that lie on the unit circle. Prove that U is a subgroup of $\mathbb{C}^*$. Then, prove or disprove: U is normal in $\mathbb{C}^*$.

13. (4 points) What was your favorite topic in this class? Specifically, what did you find the most interesting, and why?