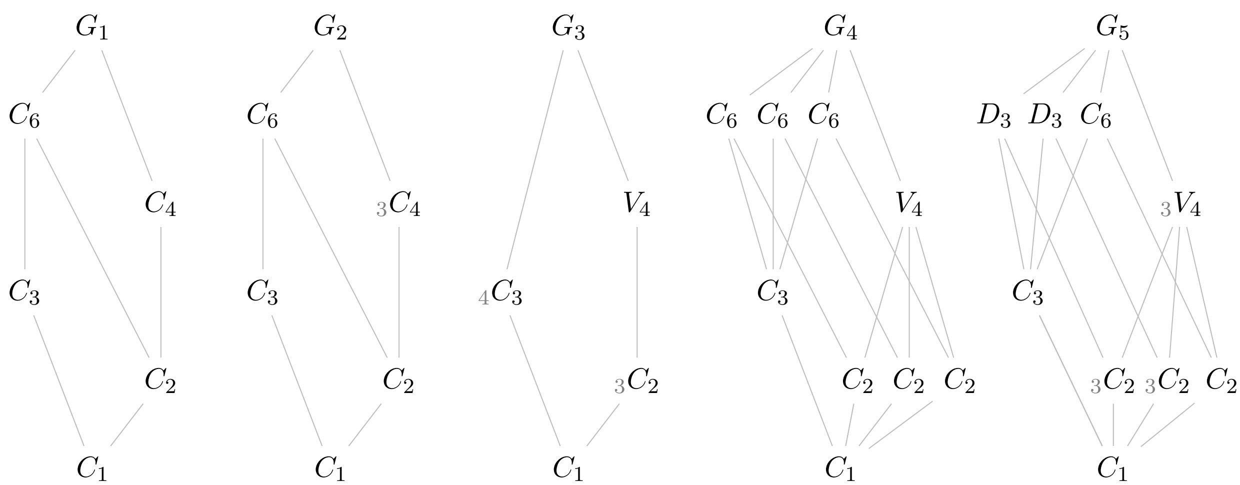
Math 4120, Midterm 1. March 4, 2025

Write your answers for these problems *directly on this paper.* You should be able to fit them in the space given.

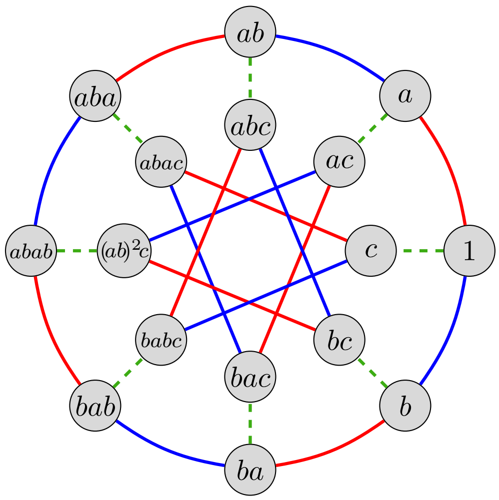
1. (15 pts) There are five groups of order twelve: C_{12}, $C_6 \times C_2$, D_6, Dic_6, and A_4. Below are the subgroup diagrams of all of them, not necessarily in that order.

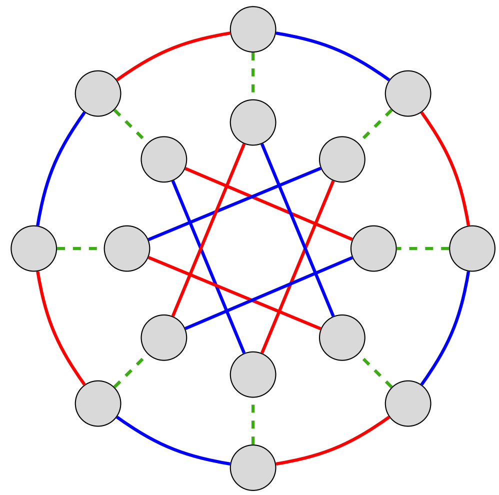


Fill out the following table.

	G_1	G_2	G_3	G_4	G_5
# subgroups					
# normal subgroups					
# conj. classes of subgps					
abelian? (Y/N)					
cyclic? (Y/N)					
$G_i \cong$					

2. (34 pts) Consider the Cayley graph of a group $G = \langle a, b, c\rangle$ shown below, with subgroups $H = \langle abab, c\rangle$ and $K = \langle ab\rangle$. The green edges are dashed just to help differentiate them from the red edges. Though this looks like the diquaternion group DQ_8, it's not!





(a) In the nodes of the graph on the right, write the *order* of the corresponding elements.

(b) The group G has order ______________, and it (is)(is not) [$\longleftarrow$ *circle one*] abelian.

(c) The subgroup $\langle a, b\rangle$ has order _________, index _________, and (is)(is not) normal.

(d) The subgroup $\langle c\rangle$ has order __________, index __________, and (is)(is not) normal.

(e) The subgroup $\langle ac\rangle$ is isomorphic to the familiar group ________________________.

(f) The subgroup $\langle a, b\rangle$ is isomorphic to the familiar group _______________________.

(g) The quotient group $G/\langle a, b\rangle$ is isomorphic to ________________________________.

(h) Find the left cosets of H. Write them as subsets. (Don't forget H itself.)

(i) Write down the right cosets of the subgroup H. Write them as subsets.

(j) Find the *normalizer* of H.

(k) Find all conjugate subgroups to H. Write each subgroup only once.

(l) Is H normal? Why or why not?

(m) Find the left cosets of the subgroup K. Write them as subsets.

(n) Write down the right cosets of the subgroup K. Write them as subsets.

(o) Find the *normalizer* of K.

(p) Find all conjugate subgroups to K. Write each subgroup only once.

(q) Is K normal? Why or why not?

(r) The subgroups $H \cong$ ____________________ and $K \cong$ ____________________.

(s) Which familiar group is G isomorphic to? Justify your answer.

3. (15 pts) Finish the following formal mathematical definitions. For full credit, you must properly use $\forall$ (for all) or $\exists$ (there exists), where appropriate.

(a) A subgroup $H \leq G$ is *normal* if ______________________________________.

(b) The *center* of a group G is $Z(G) := \Big\{ \qquad \Big\}$.

(c) The *normalizer* of a subgroup $H \leq G$ is $N_G(H) := \Big\{ \qquad \Big\}$.

(d) The *centralizer* of an element $h \in G$ is $C_G(h) := \Big\{ \qquad \Big\}$.

(e) The *conjugacy class* of an element $h \in G$ is $\mathrm{cl}_G(h) := \Big\{ \qquad \Big\}$.

4. (12 pts) Let $G = \mathrm{GL}_2(\mathbb{R})$, the 2×2 invertible matrices over the real numbers. Recall that the *special linear group* $H = \mathrm{SL}_2(\mathbb{R}) \subseteq \mathrm{GL}_2(\mathbb{R})$ consists of the those matrices that have determinant 1. Prove that H is a subgroup of G, and that it is normal. You may assume that $\det(AB) = \det(A)\det(B)$.

5. (4 pts) Write down every abelian group of order $42 = 2 \cdot 3 \cdot 7$, up to isomorphism.

6. (4 pts) Prove that if G is a noncyclic group of order $|G| = 42$, then it must be nonabelian.

7. (8 pts) Let $H \leq G$. Define the sets

$$abH = \{abh \mid h \in H\}, \qquad aHbH = \{ah_1bh_2 \mid h_1, h_2 \in H\}.$$

Prove that if H is normal in G, then $abH = aHbH$. [Show $\subseteq$ and $\supseteq$ separately.]

8. (8 pts) Draw the subgroup lattice of $S_3 = \{e, (123), (132), (12), (13), (23)\} \cong D_3$, where each subgroup is written by generator(s). Denote each conjugacy class $\mathrm{cl}_G(H)$ by circling it on the lattice. Write "$Z(G)=$" by the center, and find the normalizer of each subgroup.