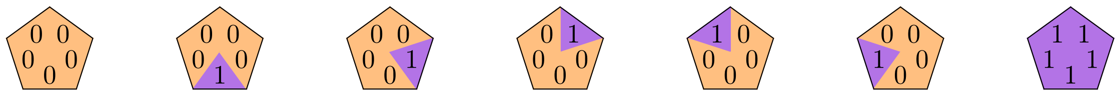
Math 4120, Midterm 2. April 15, 2025

1. (22 pts) Consider the following set S_1 of seven "binary pentagons":



The group $G = D_5 = \langle r, f \rangle$ acts on S_1 via $\phi_1 \colon G \to \text{Perm}(S_1)$, where

$\phi(r) =$ rotates each pentagon 72° counterclockwise, $\phi(f) =$ reflects each pentagon about a vertical axis.

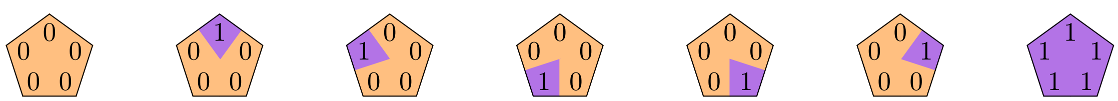
(a) Draw the *action graph*. (No need to re-draw these pentagons, just add edges to what appears above.) Distinguish r and f-paths differently, e.g., dashed or double lines, use colors, etc.

(b) Above or beside each pentagon, write down its stabilizer in terms of generator(s), i.e., as $\langle \dots \rangle$.

(c) Fill in the following. Feel free to write, e.g., $\{\#1, \#7\}$, instead of actually re-drawing the pentagons.

- $\text{fix}(f) =$
- $\text{fix}(rf) =$
- $\text{fix}(r^2 f) =$
- $\text{fix}(r) =$
- $\text{fix}(1) =$
- Average size of $\text{fix}(g)$, where $g \in D_5 =$
- $\text{Fix}(\phi) =$
- $\text{Ker}(\phi) =$

(d) Now, repeat Parts (a) and (b), for the action $\phi_2 \colon G \to \text{Perm}(S_2)$ on the following set S_2 of binary pentagons.



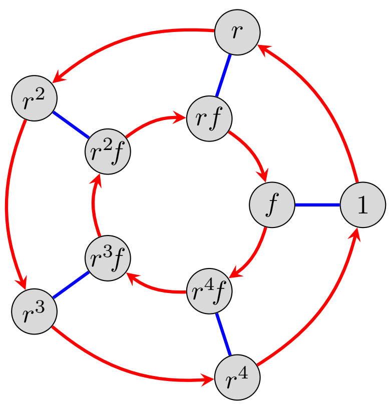
(e) Are the two actions $\phi_1 \colon G \to \text{Perm}(S_1)$ and $\phi_2 \colon G \to \text{Perm}(S_2)$ *equivalent*? If yes, describe the isomorphism $\iota \colon D_5 \to D_5$ and bijection $\sigma \colon S_1 \to S_2$ that works. If no, explain why.

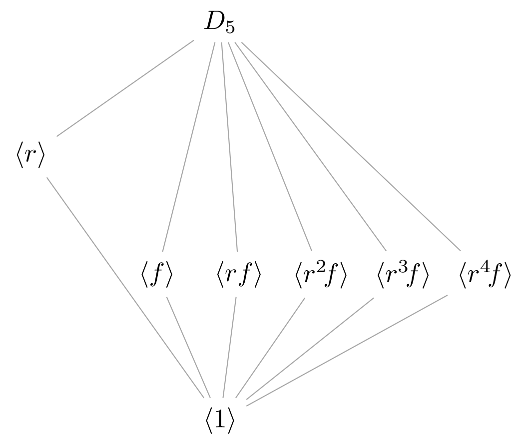
(f) Are S_1 and S_2 *isomorphic as G-sets*? Justify your answer.

(g) Describe how to construct a transitive D_5-set (i.e., the action graph is connected) consisting of exactly two "decorated" pentagons. Draw the action graph.

2. (15 pts) Consider the group $G = D_5$ whose Cayley graph and subgroup lattice are shown below.

Index **Order**





(a) For each "level" in the subgroup lattice, write the *order* and the *index* of the subgroups, under the words "**Order**" and "**Index**."

(b) Consider the (right) action of $G = D_5$ on its set of subgroups, where $\phi(g)\colon H \mapsto g^{-1}Hg$. Draw the action graph, superimposed on the subgroup lattice above. [Be careful not to do $\phi(g)\colon H \mapsto gHg^{-1}$.]

(c) For this D_5-action (on its subgroups), $\text{stab}\big(\langle rf\rangle\big) =$ ___________ and $\text{stab}\big(\langle r^2 f\rangle\big) =$ ___________.

(d) Let $G = D_5$ act on the right cosets of $H = \langle f\rangle$ by $\phi(g)\colon Hx \mapsto Hxg$. Draw the action graph, and beside each coset, write its stabilizer.

(e) How many distinct transitive (i.e., connected) D_5-sets are there, up to isomorphism? Draw an action graph for each one, using the generating set $D_5 = \langle r, f\rangle$.

3. (17 pts) Fill in the following blanks:

   (a) The action of D_5 on itself by multiplication has ________ orbit(s) and ________ fixed points(s).

   (b) The action of D_5 on its subgroups by conjugation has ______ orbit(s) and ______ fixed points(s).

   (c) The action of D_5 on the cosets of $H = \langle f \rangle$ by multiplication has __ orbit(s) and __ fixed points(s).

   (d) The group D_5 has ____________ Sylow 2-subgroup(s) and ____________ Sylow 5-subgroup(s).

   (e) Find all distinct ways (i.e., up to isomorphism) that D_5 can be written as a direct or semidirect product of two of its proper subgroups: ____.

   (f) A homomorphism $\phi\colon G \to H$ is ____________________________ if and only if $\mathrm{Ker}(\phi) = \{e\}$.

   (g) A homomorphism $\phi\colon G/N \to H$ is *well-defined* if $aN = bN$ implies ______________________.

   (h) A homomorphism $\phi\colon G/N \to H$ is *injective* if ________________ implies ________________.

   (i) Recall that an automorphism is *inner* if is has the form $g \mapsto x^{-1}gx$. The inner automorphism group of V_4 is $\mathrm{Inn}(V_4) \cong$ ____.

   (j) If G is nonabelian and $\phi\colon G \to \mathbb{Z}$ a homomorphism, then $\phi(ab) =$____________ for all $a, b \in G$.

   (k) If G is nonabelian and $\phi\colon G \to \mathbb{Z}$ a homomorphism, then $\phi(e) =$______________________.

   (l) If G is nonabelian and $\phi\colon G \to \mathbb{Z}$ a homomorphism with $\phi(g) = 1$, then $\phi(g^{-1}) =$____________.

4. (10 pts) Define a homomorphism

$$\phi\colon D_4 \longrightarrow \mathbb{Z}_2^2 = \{00, 01, 10, 11\}, \qquad \phi(r) = 10, \quad \phi(f) = 01.$$

   (a) Find the image of the following elements:

   $\phi(1) =$ $\qquad$ $\phi(r^2) =$ $\qquad$ $\phi(r^3) =$ $\qquad$ $\phi(rf) =$ $\qquad$ $\phi(r^2 f) =$ $\qquad$ $\phi(r^3 f) =$

   (b) $\mathrm{Ker}(\phi) =$____________, and $D_4/\mathrm{Ker}(\phi)$ is isomorphic to the familiar group ____________.

   (c) This homomorphism (is)(is not) [$\longleftarrow$ *circle one*] injective, and (is)(is not) surjective.

5. (12 pts) Let G be a group of order $40 = 2^3 \cdot 5$.

   (a) A Sylow 2-subgroup of G has order _________, and a Sylow 5-subgroup has order _________.

   (b) Prove that G cannot be simple. State any results that you use.

6. (24 pts) Let A and B be subgroups of G, and assume that A normalizes B (i.e., that $aB = Ba$ for every $a \in A$).

   (a) Prove that $B \trianglelefteq AB$.

   (b) Prove that $(A \cap B) \trianglelefteq A$.

   (c) By definition of a quotient group, $AB/B = \Big\{ \qquad\qquad\qquad\qquad\qquad\qquad \Big\}$.

   (d) Show (or just briefly explain why) $AB/B = A/B$.

   (e) Prove that the following quotient groups are isomorphic:

$$A/(A \cap B) \cong AB/B.$$