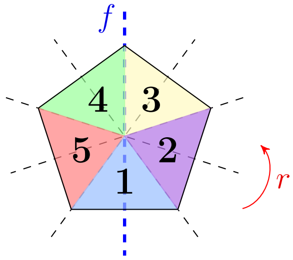
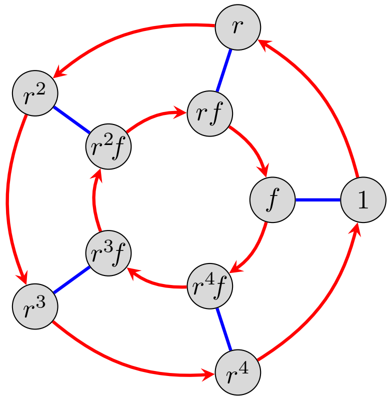
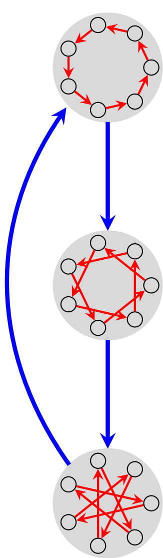
1. All of the subgroups of D_5 should be visually apparent from thinking about symmetries of a regular pentagon, shown below at left. At right is a Cayley graph.

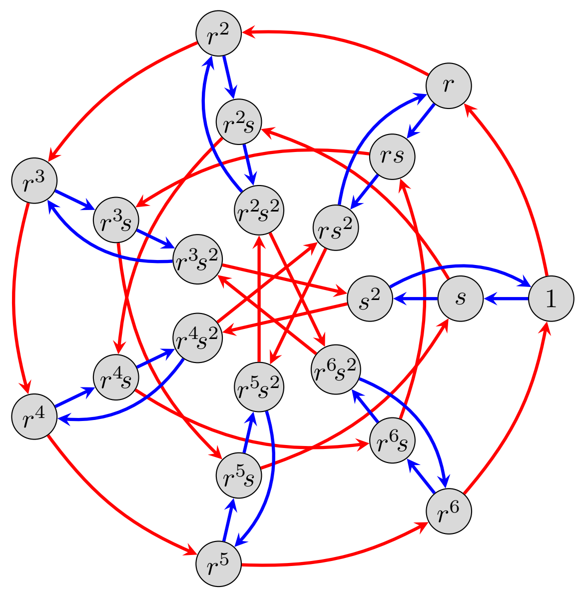




   (a) Construct a subgroup lattice for D_5. Label each edge from H to K with $[H : K]$.
   (b) Find the left and right cosets of the subgroups $\langle r\rangle$ and $\langle f\rangle$.
   (c) The *normalizer* of $H \le G$, denoted $N_G(H)$, is the union of the left cosets of H that are also right cosets. Find the normalizer of $\langle r\rangle$ and $\langle f\rangle$.
   (d) Two subgroups $H, K \le G$ are *conjugate* if $K = gHg^{-1} := \{ghg^{-1} \mid h \in H\}$ for some $g \in G$. This defines an equivalence relation on the set of subgroups called *conjugacy classes*. Partition the subgroups of D_5 into conjugacy classes.

2. Cayley graph of the smallest non-abelian group of odd order, $G = C_7 \rtimes C_3$, is shown below, highlighting its semidirect product structure.





   (a) On a blank Cayley graph, label nodes with the order of the corresponding elements. Then construct a cycle graph, labeled by group elements.
   (b) Construct a subgroup lattice and label each edge with the corresponding index.
   (c) Find the left and right cosets of the subgroups $\langle r\rangle$ and $\langle s\rangle$, and their normalizers.
   (d) Partition the subgroups into conjugacy classes, and denote this on your lattice.

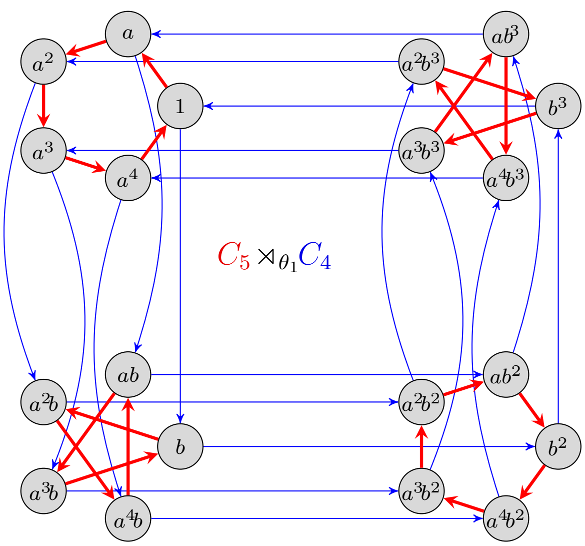
3. In this problem, you will construct the semidirect product $C_9 \rtimes C_3$. Recall that $\mathrm{Aut}(C_9)$ was constructed on the previous assignment.

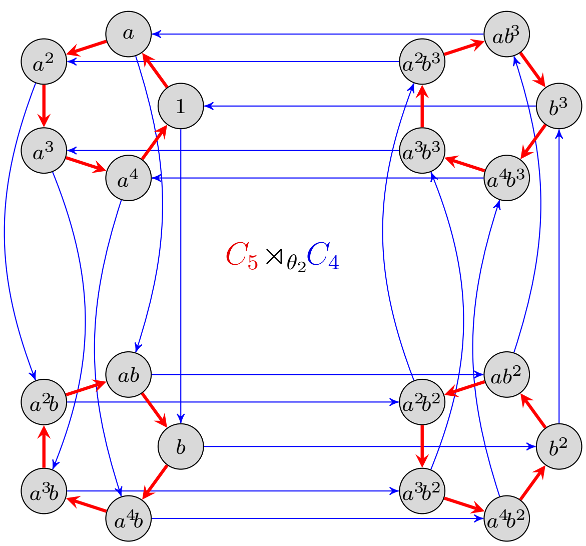
   (a) Find all possible labeling maps $\theta\colon C_3 \to \mathrm{Aut}(C_9)$.

   (b) Construct a nonabelian semidirect product of $C_9 = \langle r \rangle$ with $C_3 = \langle s \rangle$, using a labeling map that makes the Cayley graph less tangled. Include a Cayley graph of C_3 with the nodes labeled by $\theta(s^j)$, and a Cayley graph of $C_9 \rtimes_\theta C_3$, with the nodes labeled by $r^i s^j$.

   (c) Repeat Problem 2, but for the group $G = C_9 \rtimes C_3$. It is helpful to know that it has four subgroups of order 9 and four subgroups of order 3.

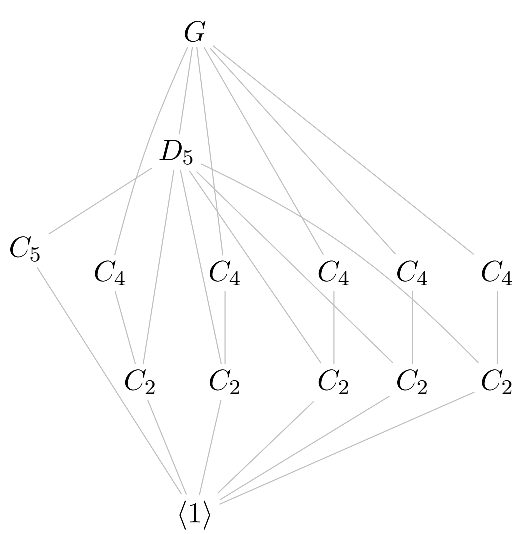
4. Consider two semidirect products of C_5 with C_4, whose Cayley graphs are shown below.

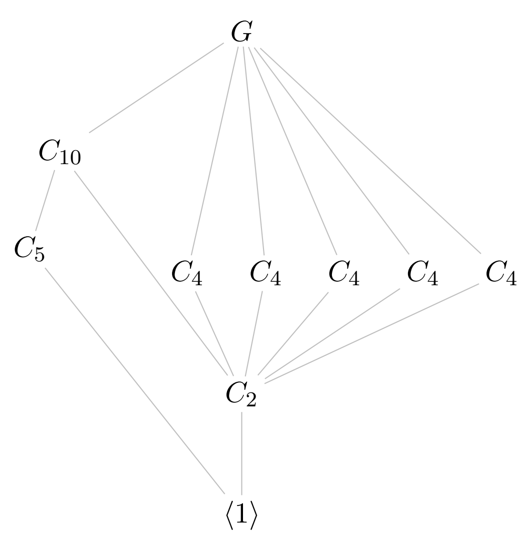




   (a) Construct a cycle graph for each group, with the nodes labeled by group elements.

   (b) The subgroup lattices of these two groups are shown below. Re-draw them with the subgroups written by generators.





(c) Determine which group each of these is isomorphic to, and which elements a and b correspond to. Recall that there are only three nonabelian groups of order 20:

$$D_{10} = \langle r, f \mid r^{10} = f^2 = 1,\, rfr = f \rangle, \qquad \mathrm{Dic}_{10} = \langle r, s \mid r^{10} = s^4 = 1,\, r^5 = s^2 \rangle,$$

$$\mathrm{AGL}_1(\mathbb{Z}_5) = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle \leq \mathrm{GL}_2(\mathbb{Z}_5).$$

Write a presentation for both groups in this problem, in terms of a and b.

(d) Construct the subgroup lattice for $G = D_{10}$. It helps to think of the subgroups geometrically—there are two subgroups isomorphic to D_5, unique cyclic subgroups of orders 10 and 5, five subgroups isomorphic to V_4, and 11 subgroups of order 2.

(e) For each of the diagrams below, determine whether it is the Cayley graph of a group. If yes, write a presentation and determine whether it is isomorphic to D_{10}, Dic_{10}, or $\mathrm{AGL}_1(\mathbb{Z}_5)$. If no, explain why.