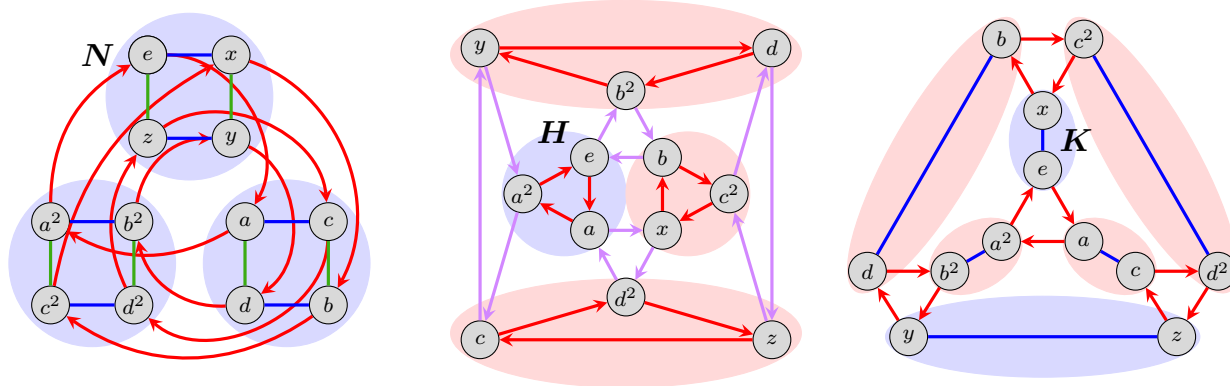
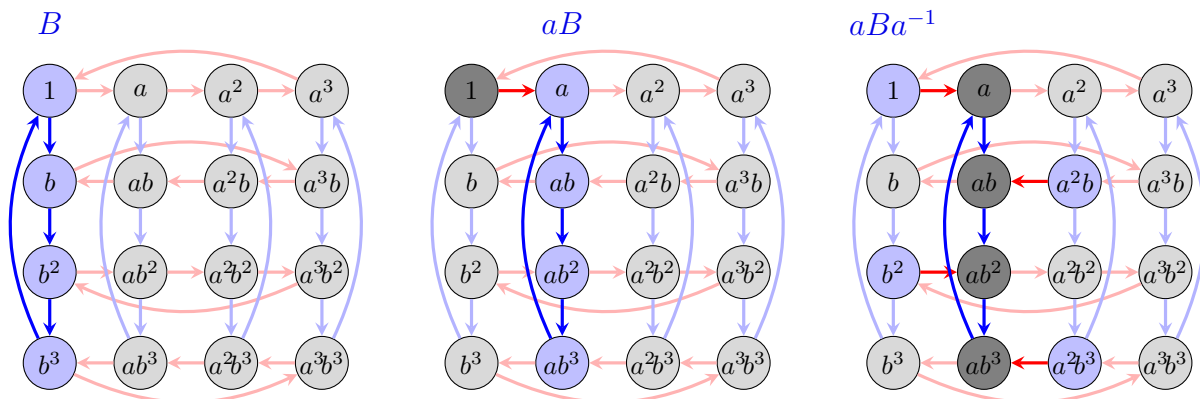


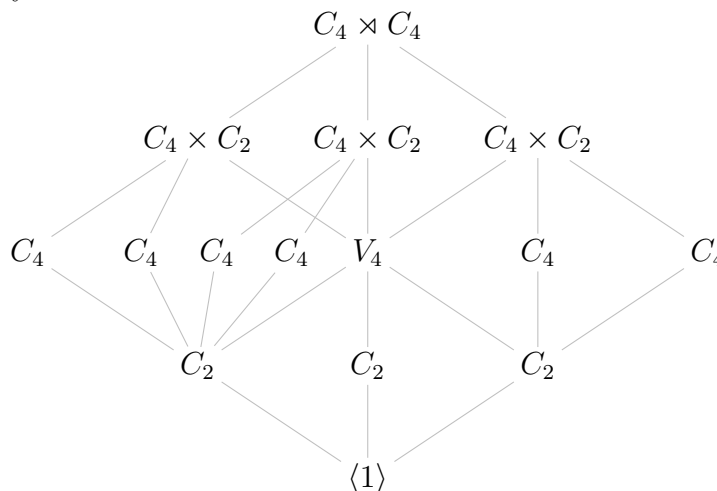
1. Below are three Cayley graphs of  $A_4$ , each highlighting the left cosets of a different subgroup. We can take  $a = (123)$ ,  $b = (134)$ ,  $x = (12)(34)$ , and  $z = (13)(24)$ .



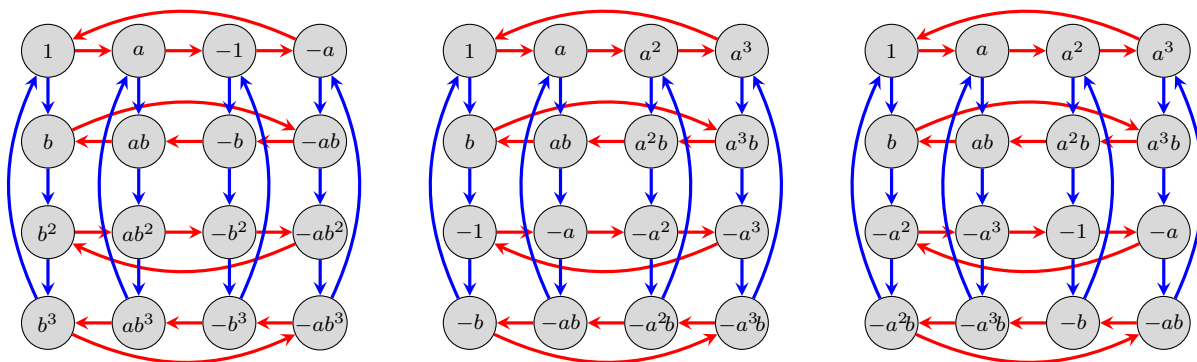
- (a) For each subgroup shown above, partition  $A_4$  into its right cosets. Write them as subsets of  $A_4$ , consisting of permutations in cycle notation. Also, highlight them by colors on a fresh copy of the Cayley graphs.
- (b) For each left coset  $gH$ , illustrate the construction of the conjugate subgroup  $gHg^{-1}$  on a fresh copy of the Cayley graph. Repeat this for  $N$  and  $K$ .
2. Show that  $A \times \{1_B\}$  is a normal subgroup of  $A \times B$ , where  $1_B$  is the identity element of  $B$ . That is, show first that it is a subgroup, and then that it is normal.
3. Let  $G$  be a group, not necessarily finite, and  $H \leq G$ .
- (a) Show that the subgroup
- $$N := \bigcap_{x \in G} xHx^{-1}$$
- is normal in  $G$ .
- (b) Show that *every* normal subgroup  $K \trianglelefteq G$  contained in  $H$  is contained in  $N$ . In other words,  $N$  is the *largest normal subgroup* of  $G$  contained in  $H$ .
4. Shown below is a Cayley graph for  $G = C_4 \rtimes C_4 = \langle a, b \rangle$ , and the construction of the conjugate subgroup  $aBa^{-1}$ , where  $B = \langle b \rangle$ .



- (a) For both order-4 subgroups,  $\langle ab \rangle$  and  $\langle a^2, b^2 \rangle$ , illustrate the construction of its conjugate subgroups on the Cayley graph, in the same 3-step process that was done above for  $B = \langle b \rangle$ . Carry this out for each of its three distinct left cosets (excluding the subgroup itself).
- (b) The subgroup lattice of  $G = \langle a, b \rangle$  is shown below. Redraw this with the subgroups written by generator(s). Then partition the subgroups into conjugacy classes, and fully justify your answer.



- (c) Without computing left or right cosets, i.e., only from the knowledge of conjugacy classes, find the normalizer of each subgroup. Justify your answer.
- (d) Construct a (labeled) cycle graph. Which subgroup is the center,  $Z(G)$ , and why?
- (e) Shown below are three copies of the Cayley graph for  $G$ , each with a peculiar labeling of the nodes. In each graph, a pair of nodes that differ by a sign represent a left coset of one of the three order-2 subgroups, all of which are normal.



For each graph, construct a Cayley table and Cayley graph consisting of these eight “cosets”. Determine what the resulting *quotient group* is isomorphic to, and describe where that subgroup lattice appears, hiding in the lattice for  $G$ .

5. Let  $H$  be a subgroup of  $G$ . Given two fixed elements  $a, b \in G$ , define the sets

$$aHbH = \{ah_1bh_2 \mid h_1, h_2 \in H\} \quad \text{and} \quad abH = \{abh \mid h \in H\}.$$

Show that if  $H \trianglelefteq G$ , then  $aHbH = abH$ .