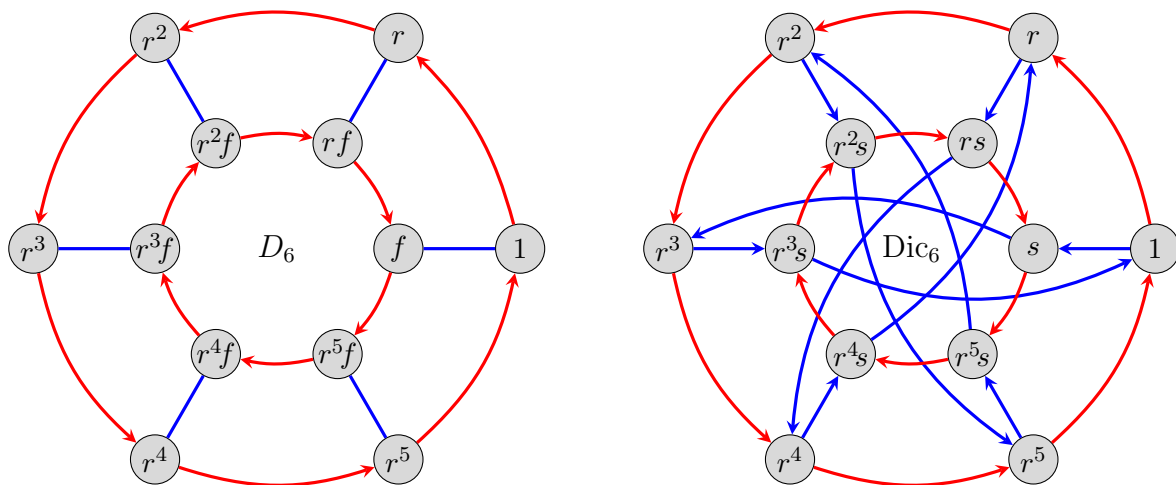


1. In both the dihedral group  $G = D_6$  and dicyclic group  $G = \text{Dic}_6$ , whose Cayley graphs are shown below, the subgroups  $N = \langle r^3 \rangle$  and  $H = \langle r^2 \rangle$  are normal. For both, construct a Cayley table and Cayley graph for the quotients  $G/N$  and  $G/H$ , and determine what these are isomorphic to.



2. Let  $H$  be a subgroup of  $G$ .
- Show that if  $G$  is abelian, then  $H$  and  $G/H$  are abelian.
  - Show that if  $G/Z(G)$  is cyclic, then  $G$  is abelian.
  - What cyclic groups can arise as a quotient  $G/Z(G)$ ? Justify your answer.
3. Let  $X$  be a subset of a group  $G$ . The *centralizer* of  $X$ , denoted  $C_G(X)$ , is the set of all elements that commute with everything in  $X$ :

$$C_G(X) = \{g \in G \mid gx = xg, \forall x \in X\}.$$

If  $X = \{x\}$ , then we denote the centralizer as  $C_G(x)$ .

- Show that  $C_G(X)$  is a subgroup of  $G$ .
- If  $H$  is a subgroup of  $G$ , show that  $C_G(H) \trianglelefteq N_G(H)$ .
- Fix  $x \in G$ , and define the map

$$\phi: \{\text{left cosets of } C_G(x)\} \longrightarrow \{\text{conjugates of } x\}, \quad \phi: gC_G(x) \longmapsto gxg^{-1}.$$

Show that this map is well-defined and a bijection.

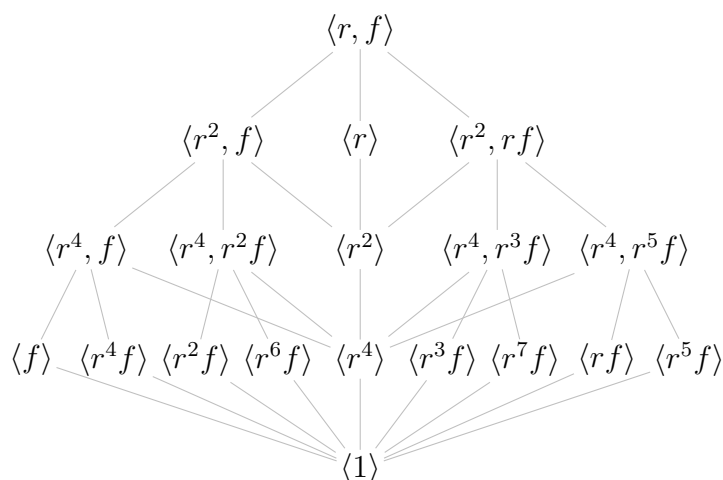
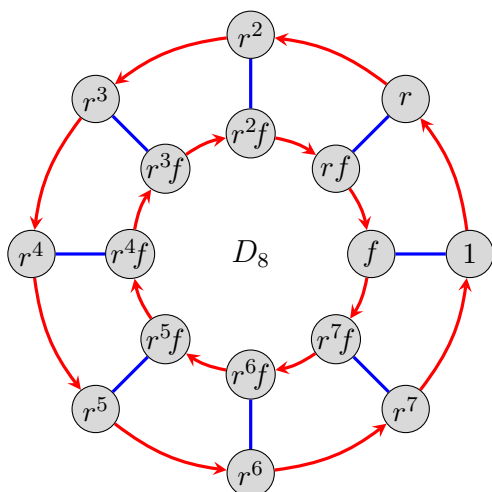
- Derive the useful formula  $|G| = |\text{cl}_G(x)| \cdot |C_G(x)|$ , for any  $x \in G$ .

4. The partition of the groups  $Q_8$  and  $D_6$  into conjugacy classes is shown below.

1	$i$	$j$	$k$
-1	$-i$	$-j$	$-k$

1	$r$	$r^2$	$f$	$r^2 f$	$r^4 f$
$r^3$	$r^5$	$r^4$	$rf$	$r^3 f$	$r^5 f$

- (a) For  $Q_8$  and  $D_6$ , compute the centralizers of each element  $x \in G$ , and the normalizer  $N_G(\langle x \rangle)$  as well.
- (b) Partition the elements of the group  $D_8$  by conjugacy classes, and arrange them in a table, as above. Then repeat the previous part for this group. The Cayley graph and subgroup lattice for  $D_8$  is shown below, for convenience.



5. Recall that two elements in  $S_n$  are conjugate if and only if they have the same cycle type.
- (a) Determine how many elements there are of each cycle type in  $S_4$ , and in  $S_5$ . Note that the sum of your answers should add up to  $|S_4| = 4! = 24$  and  $|S_5| = 5! = 120$ , respectively.
- (b) Partition the elements of  $S_4$  by conjugacy class.
- (c) Compute the centralizers of  $e$ ,  $(12)$ ,  $(123)$ ,  $(1234)$ , and  $(12)(34)$  in  $S_4$ .
- (d) Partition the elements of  $A_4$  by conjugacy class. Then pick one element from each class, and find its centralizer. [Hint: two-thirds of the elements in  $A_4$  are 3-cycles, so they cannot all be in the same conjugacy class.]
- (e) Find the centralizer of each of the elements  $e$ ,  $(12)$ ,  $(123)$ ,  $(1234)$ ,  $(12345)$ ,  $(12)(34)$ , and  $(123)(45)$  in  $S_5$ .