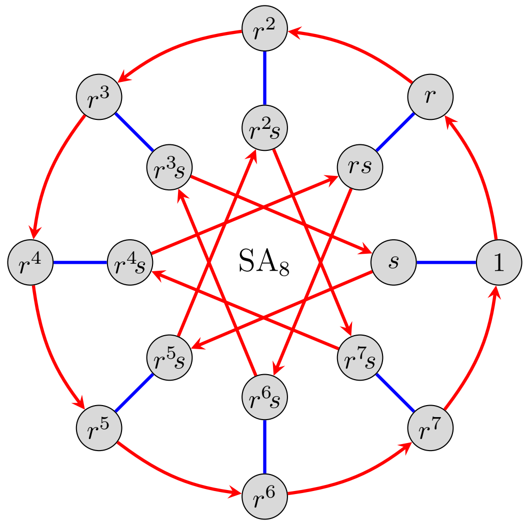
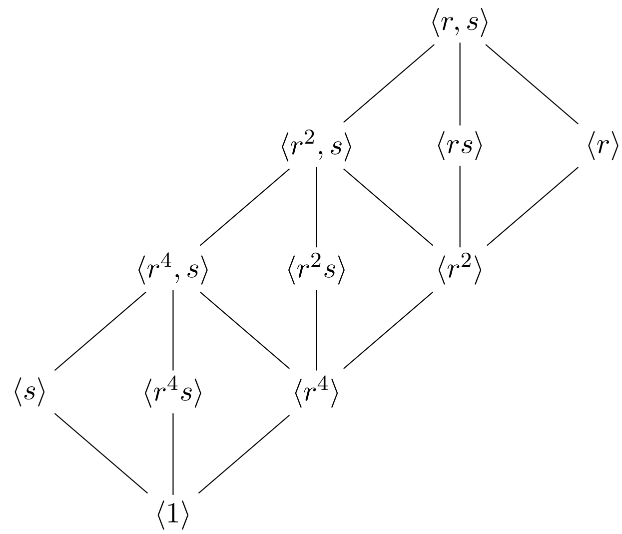
1. Let G be the *semiabelian group* of order 16, defined by the presentation

$$\mathrm{SA}_8 = \langle r, s \mid r^8 = s^2 = 1, srs = r^5 \rangle,$$

A Cayley graph and subgroup lattice are shown below.





(a) The subgroups $V = \langle r^4, s\rangle$, $H = \langle r^2 s\rangle$, $K = \langle r^2\rangle$, and $N = \langle r^4\rangle$ are all normal. Highlight their cosets on a fresh Cayley diagram by colors.

(b) Construct a Cayley table for the quotient of G by each of these subgroups. Then draw a Cayley graph for each, labeling the nodes with elements (i.e., cosets).

(c) Let $N = \langle r^4\rangle$. The shaded region below shows an order-4 cyclic subgroup of G/N, generated by the element rN, and how the union of these four cosets is the order-8 subgroup $\langle r\rangle$ of G. Construct analogous tables for the other five non-trivial proper subgroups of G/N, and then draw the subgroup lattice of G/N.

r^3N	r^3sN
r^2N	r^2sN
rN	rsN
N	sN

$\langle rN\rangle \leq G/N$

r^3	r^7	r^3s	r^7s
r^2	r^6	r^2s	r^6s
r	r^5	rs	r^5s
1	r^4	s	r^4s

$\langle r\rangle/N \leq G/N$

r^3	r^7	r^3s	r^7s
r^2	r^6	r^2s	r^6s
r	r^5	rs	r^5s
1	r^4	s	r^4s

$\langle r\rangle \leq G$

(d) For each subgroup M/N from Part (c), determine what the quotient of G/N (order 8) by M/N (order 4 or 2) is isomorphic to. Justify your answer.

(e) One step of Part (c) consisted of starting with G, taking the quotient by N, and then taking the subgroup generated by r^2N and sN. Compare and contrast this to doing these steps in the reverse order. That is: *start with G, first take the subgroup $\langle r^2, s\rangle$, and then take the quotient by N.*

(f) Repeat Part (c) for subgroups $V = \langle r^4, s\rangle$, $H = \langle r^2 s\rangle$, and $K = \langle r^2\rangle$ of G. This time, include the trivial and proper subgroups for each.

2. Show that there is no embedding $\phi\colon \mathbb{Z}_n \hookrightarrow \mathbb{Z}$.

3. All of the following can be done by defining an explicit map, showing that it is a homomorphism, and a bijection.

   (a) Show that $A \times B \cong B \times A$.

   (b) Show that $xHx^{-1} \cong H$, for any $H \leq G$. Conclude that $|xy| = |yx|$ for any $x, y \in G$.

   (c) Show that $(\mathbb{Q}^*, \cdot) \cong (\mathbb{Q}^+, \cdot) \times C_2$. Recall that $\mathbb{Q}^*$ and $\mathbb{Q}^+$ are the nonzero and positive rational numbers, respectively, and $C_2 = \{1, -1\}$.

4. Let $\phi\colon G \to H$ be a homomorphism, and $N \trianglelefteq H$.

   (a) Show that the set $\phi^{-1}(N) := \big\{g \in G \mid \phi(g) \in N\big\}$ is a subgroup of G.

   (b) Show that $\phi^{-1}(N)$ is a *normal* subgroup of G.

   (c) Show by example that if $M \trianglelefteq G$, then $\phi(M)$ need not be a normal subgroup of H.

5. In this exercise, you will show that if A and B are normal subgroups and $AB = G$, then

$$G/(A \cap B) \cong (G/A) \times (G/B).$$

   (a) Consider the following map:

$$\phi\colon AB \longrightarrow (G/A) \times (G/B)\,, \qquad \phi(g) = (gA, gB).$$

   Show that ϕ is a homomorphism.

   (b) Show that ϕ is surjective. That is, given any (g_1A, g_2B), show that there is some $g = ab \in AB$ such that $\phi(g) = (g_1A, g_2B)$. [*Hint*: Try $g = a_2b_1$; show this works.]

   (c) Find $\mathrm{Ker}(\phi)$ [you need to verify your answer is correct] and then apply the fundamental homomorphism theorem.