1. Let $A, B \leq G$ with A normalizing B (that is, $A \leq N_G(B)$, which implies that $AB \leq G$).
   (a) Show that $B \trianglelefteq AB$ and $A \cap B \trianglelefteq A$.
   (b) Show that $A/(A \cap B) \cong AB/B$. [*Hint:* Construct a homomorphism $\phi\colon A \to AB/B$ that has $A \cap B$ as kernel, then apply the FHT.]

2. Recall that the *commutator subgroup* is defined as

$$G' = \langle xyx^{-1}y^{-1} \mid x, y \in G \rangle.$$

   (a) Show that G' is the intersection of all normal subgroups of G that contain the set $C := \{aba^{-1}b^{-1} \mid a, b \in G\}$:

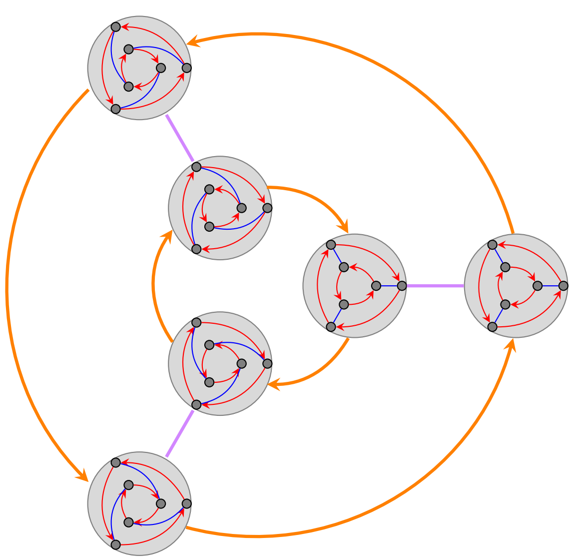
$$G' = \bigcap_{C \subseteq N \trianglelefteq G} N.$$

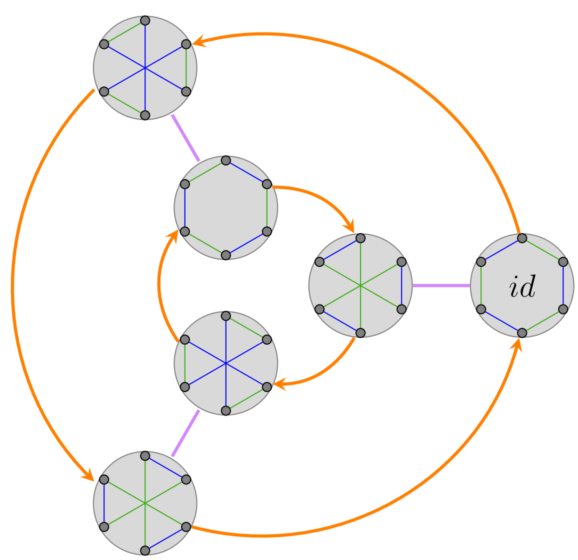
   (b) Show that G/G' is abelian. [*Hint*: Show that every commutator is trivial.]
   (c) For the groups $\mathrm{AGL}_1(\mathbb{Z}_5)$, Dic_{10}, $\mathrm{SL}_2(\mathbb{Z}_3)$, and $Q_8 \rtimes C_9$, whose subgroup lattices are shown on the supplemental material, carry out the following steps. Because all of this can be done by inspection, you need to briefly justify your answers.
      i. Partition the subgroups into conjugacy classes, by drawing dashed circles on the lattices.
      ii. The *derived series* of a group is defined as $G^{(0)} := G$, $G^{(1)} := G'$, $G^{(2)} := G''$, and inductively, $G^{(k)}$ is the commutator of $G^{(k-1)}$. Mark these groups on the lattice until the trivial group is reached, and determine the quotient $G^{(i)}/G^{(i+1)}$ of each successive pair.

3. Recall that the automorphism group of D_3 is $\mathrm{Aut}(D_3) = \langle \alpha, \beta \rangle \cong D_3$, where

$$\begin{cases} \alpha(r) = r \\ \alpha(f) = rf \end{cases} \qquad \begin{cases} \beta(r) = r^2 \\ \beta(f) = f \end{cases}$$

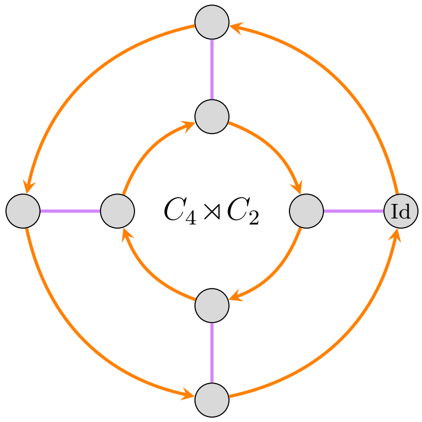
   All of these automorphisms are *inner* (of the form $\varphi_x\colon g \mapsto x^{-1}gx$). Two Cayley graphs for $\mathrm{Aut}(D_3)$ are shown below.

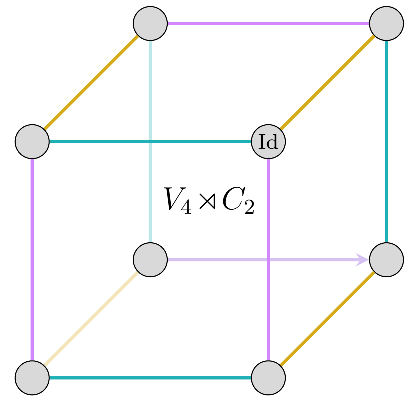




In this problem, we will construct analogous Cayley graphs for $\text{Aut}(D_4) \cong D_4$.

(a) For each of the Cayley graphs of $\text{Aut}(D_4)$ shown below, label the nodes with rewired copies of the Cayley graph of $D_4 = \langle r, f \rangle$.





(b) Repeat the previous part using the Cayley graph of $D_4 = \langle s, t \rangle = \langle f, rf \rangle$.

(c) Four of the eight automorphisms of D_4 are *inner*, which means they have the form $\varphi_x \colon g \mapsto x^{-1}gx$ for some $x \in D_4$. In fact, the automorphism group of D_4 is isomorphic to the semidirect product

$$\text{Aut}(D_4) \cong \text{Inn}(D_4) \rtimes C_2$$

of the *inner automorphism group*

$$\text{Inn}(D_4) = \big\{ \text{Id}, \varphi_r, \varphi_f, \varphi_{rf} \big\} \cong D_4/Z(D_4) = D_4/\langle r^2 \rangle \cong V_4$$

and the cyclic subgroup generated by an *outer automorphism* of order 2. In each of your Cayley graphs from Parts (a) and (b), label the nodes by the corresponding automorphism written as

$$\text{Aut}(D_4) = \big\{ \text{Id},\, \varphi_r,\, \varphi_f,\, \varphi_{rf},\, \omega,\, \varphi_r\omega,\, \varphi_f\omega,\, \varphi_{rf}\omega \big\},$$

where ω is the outer automorphism

$$\omega \colon D_4 \longrightarrow D_4, \qquad \alpha(r) = r, \quad \alpha(f) = rf$$

of order 4 that cyclically rotates axes of reflections of the square.

4. Construct each of the semidirect products below via our "inflation process".

(a) $D_3 \times C_2$ (b) $D_3 \rtimes C_2$ (c) $V_4 \rtimes C_3$ (d) $C_3 \rtimes V_4$.

Make sure you define the labeling maps $\theta \colon B \to \text{Aut}(A)$. Then determine, with justification, what each group is isomorphic to. Use $D_3 = \langle a, b \mid a^2 = b^2 = (ab)^3 = 1 \rangle$, $V_4 = \langle a, b \mid a^2 = b^2 = 1 \rangle$, and $C_n = \langle c \mid c^n = 1 \rangle$ for the individual factors, and Cayley graphs corresponding to these generating sets.