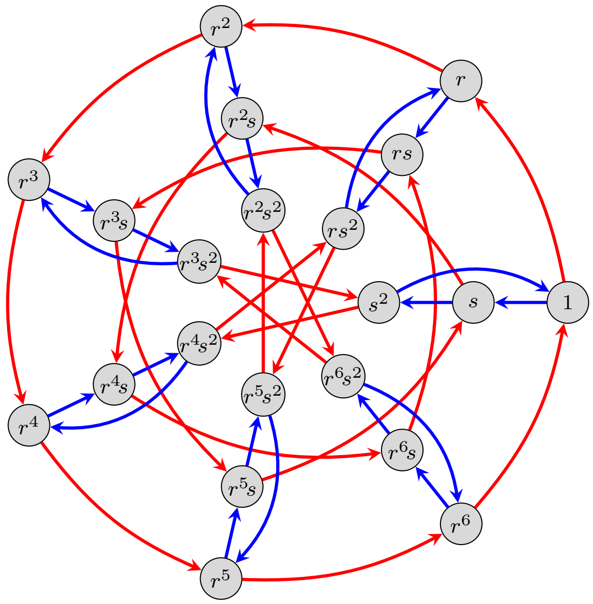
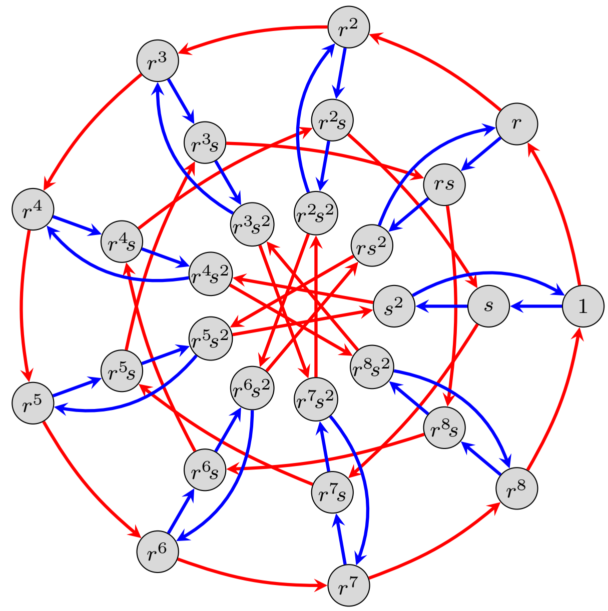
1. Carry out the following steps for the groups $C_7 \rtimes C_3$ and $C_9 \rtimes C_3$, whose Cayley graphs are shown below.





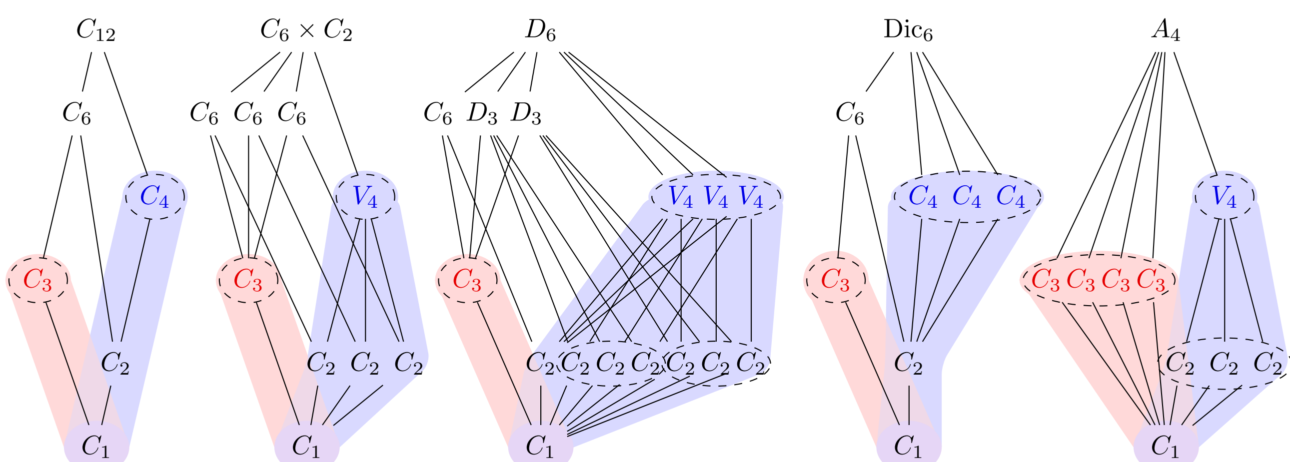
(a) Let G act on its subgroups by conjugation. Draw the action graph superimposed on the subgroup lattice. Find $\mathrm{stab}(H)$ for each $H \leq G$, $\mathrm{Ker}(\phi)$, and $\mathrm{Fix}(\phi)$.

(b) Let G act on the right cosets of $H = \langle s \rangle$, via the homomorphism

$$\phi\colon G \longrightarrow \mathrm{Perm}(S)\,, \qquad \phi(g) = \text{the permutation that sends each } Hx \mapsto Hxg.$$

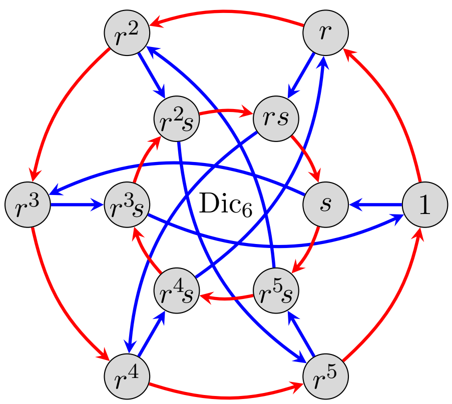
Construct the action graph. Find $\mathrm{stab}(Hx)$ for each right coset, $\mathrm{Ker}(\phi)$, and $\mathrm{Fix}(\phi)$.

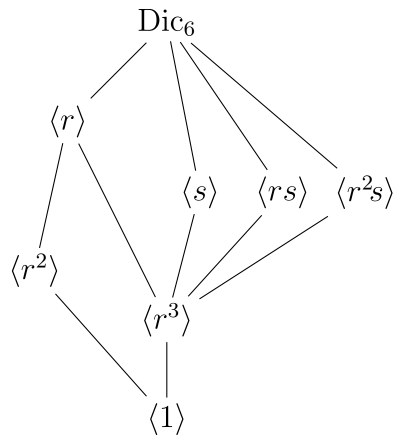
2. Loosely speaking, the upcoming Sylow theorems will us that (1) all p-subgroups come in a single "p-subgroup tower", (2) the "top" of these towers are a single conjugacy class, and (3) the size of this class is 1 mod p. This is illustrated below with the groups of order 12.



Using the LMFDB, construct analogous diagrams for the groups of order 18 and 20.

3. In this problem, we will explore the actions of the dicyclic group Dic_6 and its automorphism group on itself and its subgroups by conjugation. A Cayley graph, subgroup lattice, and conjugacy classes are shown below.





rs	r^5s	r^3s
s	r^2s	r^4s
r^3	r^2	r^4
1	r	r^5

(a) The right action of Dic_6 on itself by conjugation is defined by the homomorphism

$$\phi\colon\ \text{Dic}_6 \longrightarrow \text{Perm}(S)\,,\qquad \phi(g) = \text{the permutation that sends each } x \mapsto g^{-1}xg.$$

Draw the action graph and construct the fixed point table. Find $\text{stab}(s)$ for each $s \in S$, $\text{fix}(g)$ for each $g \in G$, as well as $\text{Ker}(\phi)$ and $\text{Fix}(\phi)$.

(b) The automorphism group of Dic_6 is $\text{Aut}(\text{Dic}_6) = \langle \varphi_r, \varphi_s, \omega\rangle$ acts on Dic_6, where ω is the outer automorphism defined by

$$\omega\colon\ \text{Dic}_6 \longrightarrow \text{Dic}_6,\qquad \omega(r) = r,\quad \omega(s) = s^{-1} = r^3s,$$

that "reverses" the blue arrows. Make a diagram showing how each automorphism permutes the elements of Dic_6. Then construct the action graph, fixed point table, and find $\text{stab}(s)$, $\text{fix}(g)$, $\text{Ker}(\phi)$ and $\text{Fix}(\phi)$.

(c) The automorphism group $\text{Aut}(\text{Dic}_6) = \langle \varphi_r, \varphi_s, \omega\rangle$ is isomorphic to D_6. Construct a Cayley graph and subgroup lattice using these generators.

(d) The group $\text{Aut}(\text{Dic}_6)$ also acts on the conjugacy classes of Dic_6. Construct the action graph, fixed point table, and find $\text{stab}(s)$, $\text{fix}(g)$, $\text{Ker}(\phi)$ and $\text{Fix}(\phi)$.

4. Let $\phi\colon G \to \text{Perm}(S)$ be a *left* group action. Prove the orbit-stabilizer theorem by constructing a bijection between $\text{orb}(s)$ and *left* cosets of $H = \text{stab}(s)$. Use analogous notational conventions from lecture, e.g., $\phi(g).s$ instead of $s.\phi(g)$.

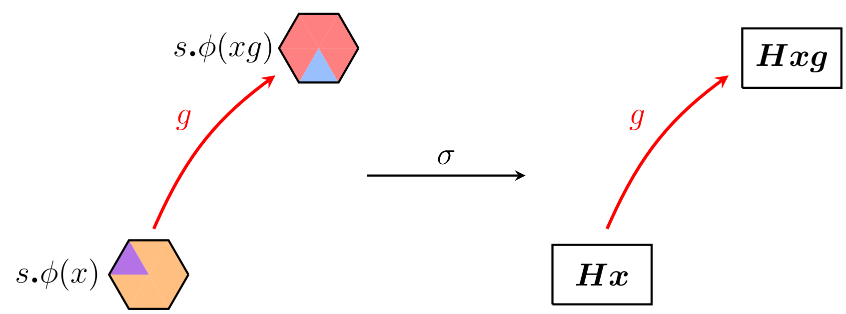
5. Recall that two group actions $\phi_1 \colon G_1 \longrightarrow \mathrm{Perm}(S_1)$ and $\phi_2 \colon G_2 \longrightarrow \mathrm{Perm}(S_2)$ are *equivalent* if there is an isomorphism $\iota \colon G_1 \to G_2$ and a bijection $\sigma \colon S_1 \to S_2$ such that $\sigma \circ \phi_1(g) = \phi_2(\iota(g)) \circ \sigma$ for all $g \in G_1$. In other words, the following diagram commutes:

$$\begin{tikzcd} S_1 \arrow[r, "\phi_1(g)"] \arrow[d, "\sigma"'] & S_1 \arrow[d, "\sigma"] \\ S_2 \arrow[r, "\phi_2(\iota(g))"'] & S_2 \end{tikzcd}$$

In class, we proved that every transitive G-action is equivalent to G acting on a set of cosets by multiplication. We did this by using the identity isomorphism $\iota \colon G \to G$ between the groups, and the bijection $\sigma \colon S \to H\backslash G$ defined by

$$\sigma \colon S \longrightarrow H\backslash G, \qquad \sigma \colon s.\phi(x) \mapsto Hx.$$

This was inpired by the following picture:



Prove an analogous statement: that every simply transitive action is equivalent to G acting on itself by conjugation. The proof is not much different than just replacing each "H" with the identity element, "1" in the one we did in class.