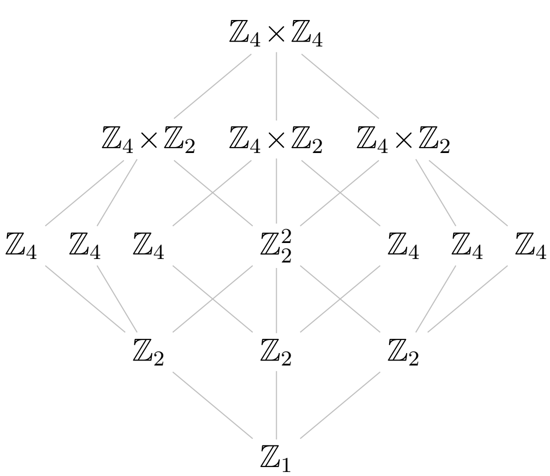
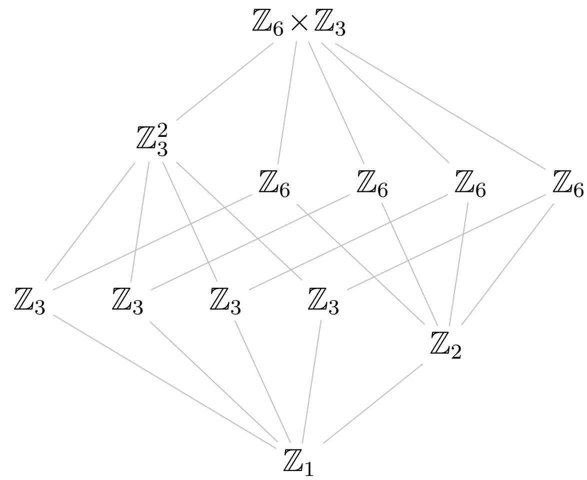
1. The subgroup lattices of $\mathbb{Z}_4 \times \mathbb{Z}_4$ and $\mathbb{Z}_6 \times \mathbb{Z}_3$ are shown below. Re-draw these lattices with the subgroups written with generator(s), and then construct their *subring lattices* by coloring each subgroup based on whether it is an ideal, subring but not an ideal, or subgroup that is not a subring.





Finally, for each ring, write down the units, and the zero divisors.

2. Let I and J be ideals of a commutative ring R.

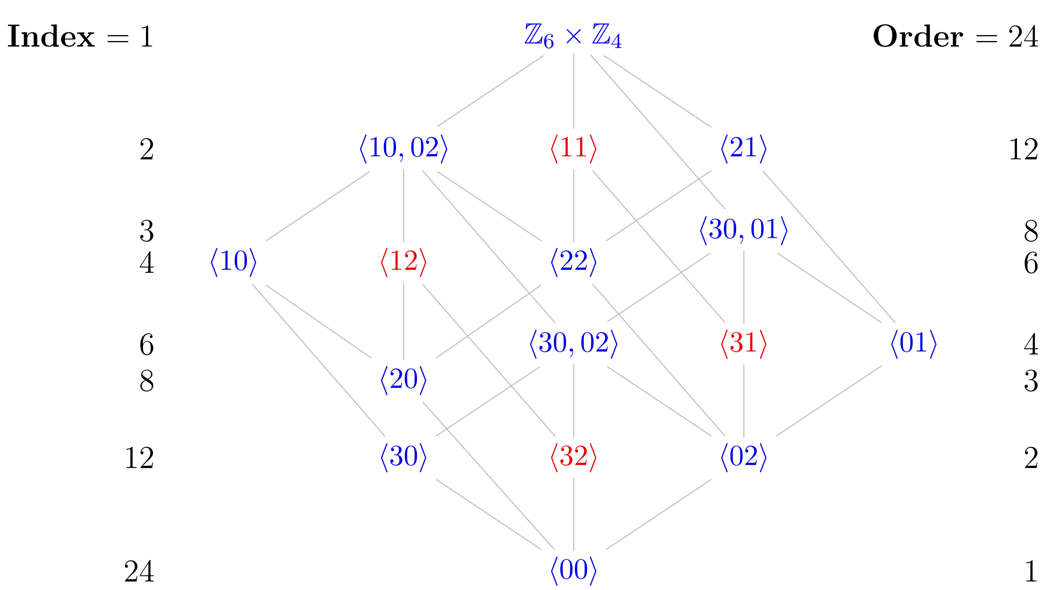
   (a) Show that $I + J$, $I \cap J$, and IJ are ideals of R. Which of these remain ideals if the commutativity hypothesis is dropped?

   (b) The set $(I : J) := \{r \in R \mid rJ \subseteq I\}$ is called the *ideal quotient* or *colon ideal* of I and J. Show that $(I : J)$ is an ideal of R. Does this require commutativity?

   (c) Determine $I + J$, $I \cap J$, IJ, and $(I : J)$ for the ideals $I = n\mathbb{Z}$ and $J = m\mathbb{Z}$ of $R = \mathbb{Z}$.

   (d) Repeat Part (c) for several pairs of ideals of $\mathbb{Z}_6 \times \mathbb{Z}_4$; see the subring lattice below.

   (e) Describe how to find IJ and $(I : J)$ by inspection, from the lattice, if possible.



3. Let $f\colon R \to S$ be a ring homomorphism between commutative rings.
   (a) If f is surjective and I is an ideal of R, show that $f(I)$ is an ideal of S.
   (b) Show that Part (a) is not true in general when f is not surjective.
   (c) Show that if f is surjective and R is a field, then S is a field as well.