Chapter 5: Actions of groups

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Overview

Intuitively, a group action occurs when a group G "naturally permutes" a set S *of states*.

For example:

- The "Rubik's cube group" consists of the 4.3×10^{19} actions that *permute* the 4.3×10^{19} configurations of the cube.
- The group D_4 consists of the 8 symmetries of the square. These symmetries are *actions* that *permute* the 8 configurations of the square.

Group actions formalize the interplay between the actual group of actions and the sets of objects that they "rearrange."

There are many other examples of groups that "act on" sets of objects. We will see examples when the group and the set have different sizes.

The rich theory of group actions can be used to prove many deep results in group theory.

We have actually already seen many group actions, without knowing it, such as:

- groups acting on themselves by multiplication
- groups acting on themselves by conjugation
- groups acting on their subgroups by conjugation
- groups acting on cosets by multiplication
- automorphism groups acting on groups.

Actions vs. configurations

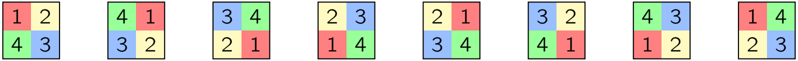
The group D_4 can be thought of as the 8 symmetries of the square:

There is a subtle but *important* distinction to make, between the actual 8 symmetries of the square, and the 8 configurations.

For example, the 8 symmetries (alternatively, "actions") can be thought of as

$$1, \quad r, \quad r^2, \quad r^3, \quad f, \quad rf, \quad r^2f, \quad r^3f\,.$$

The 8 configurations (or *states*) of the square are the following:



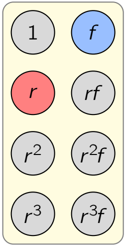
When we were just learning about groups, we made an action graph.

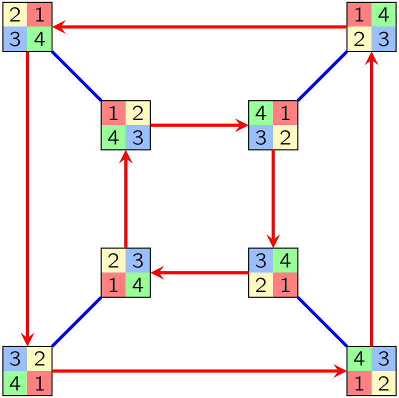
- The vertices corresponded to the states.
- The edges corresponded to generators.
- The paths corresponded to actions (group elements).

Action graphs

Here is the action graph of the group $D_4 = \langle r, f \rangle$:

"Group switchboard"





In the beginning of this course, we picked a configuration to be the "solved state," and this gave us a *bijection* between configurations and actions (group elements).

The resulting graph was a Cayley graph. In this section, we'll loosen this condition.

Action graphs

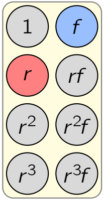
In all of the examples we saw in the beginning of the course, we had a bijective correspondence between actions and states. *This need not always happen!*

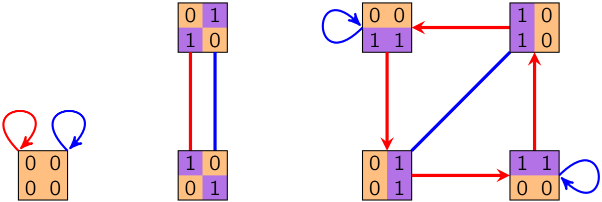
Suppose we have a size-7 set consisting of the following "binary squares."

$$S = \left\{ \begin{array}{|cc|}\hline 0 & 0 \\ 0 & 0 \\ \hline\end{array}\ ,\ \begin{array}{|cc|}\hline 0 & 1 \\ 1 & 0 \\ \hline\end{array}\ ,\ \begin{array}{|cc|}\hline 1 & 0 \\ 0 & 1 \\ \hline\end{array}\ ,\ \begin{array}{|cc|}\hline 1 & 1 \\ 0 & 0 \\ \hline\end{array}\ ,\ \begin{array}{|cc|}\hline 0 & 1 \\ 0 & 1 \\ \hline\end{array}\ ,\ \begin{array}{|cc|}\hline 0 & 0 \\ 1 & 1 \\ \hline\end{array}\ ,\ \begin{array}{|cc|}\hline 1 & 0 \\ 1 & 0 \\ \hline\end{array} \right\}$$

The group $D_4 = \langle r, f \rangle$ "acts on S" as follows:

"Group switchboard"





The action graph above has some properties of Cayley graphs, but there are some fundamental differences as well.

The “group switchboard” analogy

Suppose we have a “switchboard” for G, with every element $g \in G$ having a “button.”

If $a \in G$, then pressing the a-button rearranges the objects in S—it is a permutation of S; call it $\phi(a)$.

If $b \in G$, then pressing the b-button also rearranges the objects in S. Call this permutation $\phi(b)$.

The element $ab \in G$ also has a button. We require that pressing the ab-button does the same as pressing the a-button, followed by the b-button. That is,

$$\phi(ab) = \phi(a)\phi(b)\,, \qquad \text{for all } a, b \in G\,.$$

Let $\mathbf{Perm}(S)$ be the group of permutations of S. Thus, if $|S| = n$, then $\mathbf{Perm}(S) \cong S_n$. (We typically think of S_n as the permutations of $\{1, 2, \ldots, n\}$.)

Definition

A group G acts on a set S if there is a homomorphism $\phi\colon G \to \mathbf{Perm}(S)$.

Action graphs vs. G-sets

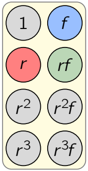
Definition

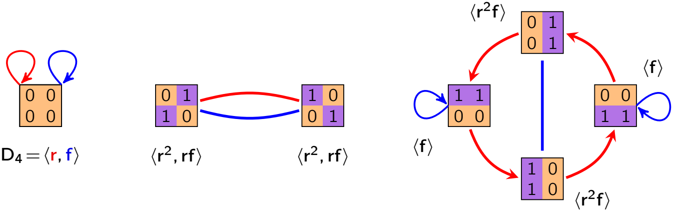
A set S with an action by G is called a (right) *G-set*.

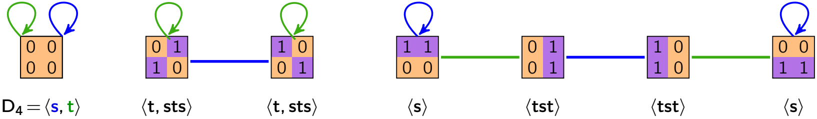
Big ideas

- An action $\phi\colon G \to \mathrm{Perm}(S)$ endows S with an **algebraic structure**.
- *Action graphs are to G-sets, like how Cayley graphs are to groups.*

"Group switchboard"

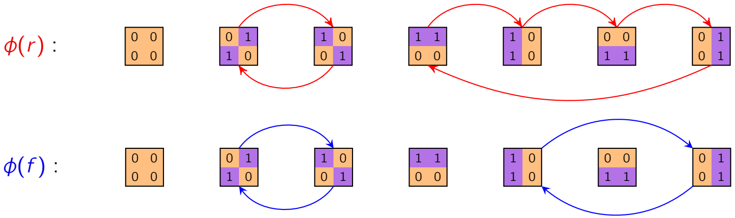






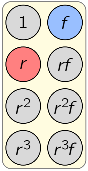
The “group switchboard” analogy

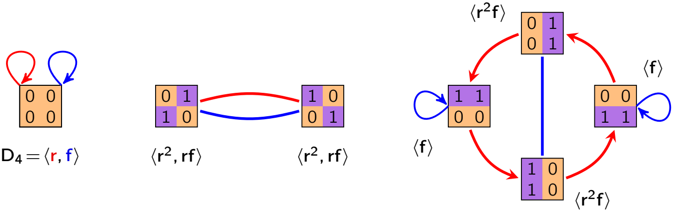
In our binary square example, pressing the r-button and f-button permutes S as follows:



Observe how these permutations are encoded in the action graph. (Next to each $s \in S$ is the subgroup that fixes it.)

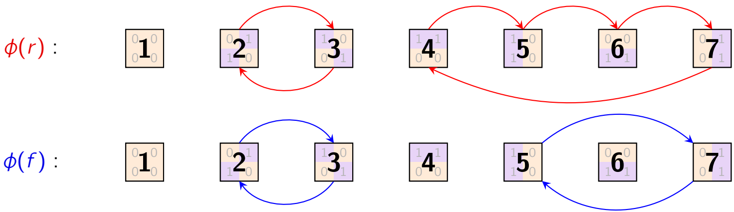
“Group switchboard”





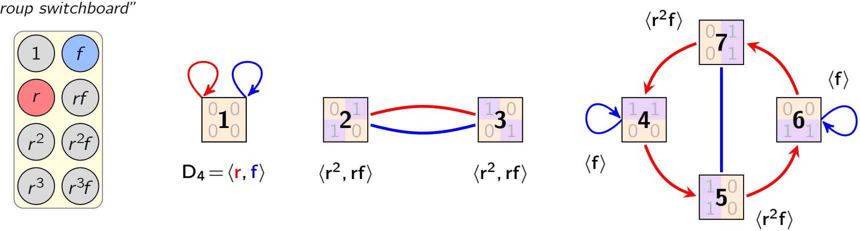
The “group switchboard” analogy

This action is an embedding $\phi\colon D_4 \hookrightarrow \mathrm{Perm}(S) \cong S_7$.



Notice that $\mathrm{Im}(\phi) = \big\langle (23)(4567), (23)(57) \big\rangle \cong D_4 \leq S_7$.

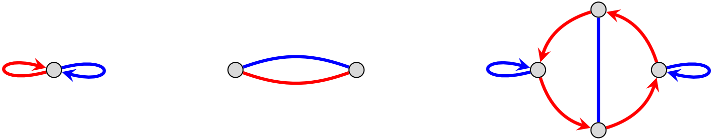
“Group switchboard”



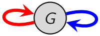
Transitive G-sets

It is natural to want to characterize all G-sets for a fixed group.

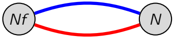
It suffices to consider all connected components. These are called transitive G-sets.



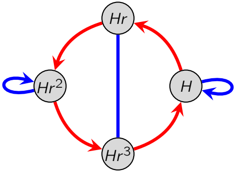
Later, we'll learn that every transitive G-set can be constructed by collapsing a Cayley graph by the cosets of some subgroup.



$G = D_4$



$N = \langle r^2, rf \rangle$



$H = \langle f \rangle$

Sometimes, action graphs are called (Schreier) coset graphs.

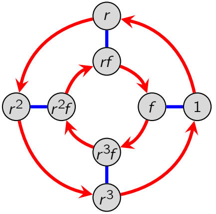
G-sets generalize groups. Action graphs generalize Cayley graphs

The group $G = D_4 = \langle r, f \rangle$ can act on itself ($S = D_4$), or on its subgroups,

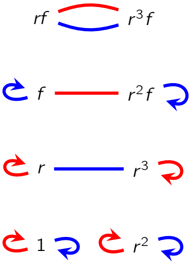
$$S = \big\{ D_4, \langle r \rangle, \langle r^2, f \rangle, \langle r^2, rf \rangle, \langle f \rangle, \langle rf \rangle, \langle r^2 f \rangle, \langle r^3 f \rangle, \langle r^2 \rangle, \langle 1 \rangle \big\}.$$

There are several ways to define the result of *"pressing the g-button on our switchboard"*.

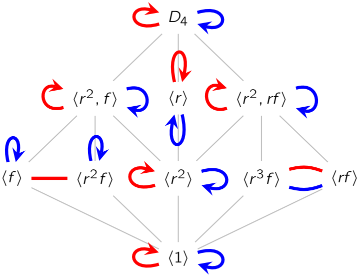
We say that: "*G acts on...*"



"*. . . itself by right-multiplication*"



"*. . . itself by conjugation*"



"*. . . its subgroups by conjugation*"

Big idea

Every Cayley graph is the action graph of a particular group action.

Left actions vs. right actions (an annoyance we can deal with)

As we've defined group actions, "*pressing the a-button followed by the b-button should be the same as pressing the ab-button*."

However, sometimes it appears like it's the same as "*pressing the ba-button*."

This is best seen by an example. Suppose our action is conjugation:

"Left group action"

$$\begin{tikzcd} H \arrow[r, "\text{conjugate by } a"] \arrow[rr, bend right, "\text{conjugate by } ba"] & aHa^{-1} \arrow[r, "\text{conjugate by } b"] & baHa^{-1}b^{-1} \end{tikzcd}$$

$$\phi(a)\phi(b) = \phi(ba)$$

"Right group action"

$$\begin{tikzcd} H \arrow[r, "\text{conjugate by } a"] \arrow[rr, bend right, "\text{conjugate by } ab"] & a^{-1}Ha \arrow[r, "\text{conjugate by } b"] & b^{-1}a^{-1}Hab \end{tikzcd}$$

$$\phi(a)\phi(b) = \phi(ab)$$

We'll call aHa^{-1} the left conjugate of H by a, and $a^{-1}Ha$ the right conjugate.

Some books forgo our "ϕ-notation" and use the following notation to distinguish left vs. right group actions:

$$g.(h.s) = (gh).s, \qquad (s.g).h = s.(gh).$$

We'll usually keep the ϕ, and write $\phi(g)\phi(h)s = \phi(gh)s$ and $s.\phi(g)\phi(h) = s.\phi(gh)$. As with groups, the "dot" will be optional.

Left actions vs. right actions (an annoyance we can deal with)

Alternative definition (other textbooks)

A right group action is a mapping

$$G \times S \longrightarrow S\,, \qquad (a,s) \longmapsto s.a$$

such that

- $s.(ab) = (s.a).b$, for all $a, b \in G$ and $s \in S$
- $s.e = s$, for all $s \in S$.

A left group action is defined similarly. Theorems for left actions have analogues for right actions.

Each left action has a related right action, and vice-versa. We'll use right actions, and write

$$s.\phi(g)$$

for "*the element of S that the permutation $\phi(g)$ sends s to*," i.e., where pressing the g-button sends s.

If we have a left action, we'll write $\phi(g).s$.

If needed, we can distinguish left G-sets from right G-sets.

Five features of every group action

Every group action has **five fundamental features** that we will always try to understand.

There are several ways to classify them. For example:

- three are subsets of S
- two are subgroups of G.

Another way to distinguish them is by local vs. global:

- three are features of individual group or set elements (we'll write in *lowercase*)
- two are features of the homomorphism ϕ. (we'll write in *Uppercase*)

We will see parallels within and between these classes.

For example, two "local" features will be "dual" to each other, as will the global features.

Our global features can be expressed as intersections of our local features, either ranging over all $s \in S$, or over all $g \in G$.

We'll start by exploring the three local features.

Notation

Throughout, we'll denote identity elements by $1 \in G$ and $e \in \mathrm{Perm}(S)$.

Two local features: orbits and stabilizers

Suppose G acts on a set S, and pick some $s \in S$. We can ask two questions about it:

(i) What other states (in S) are reachable from s? (We call this the orbit of s.)

(ii) What group elements (in G) fix s? (We call this the stabilizer of s.)

Definition

Suppose that G acts on a set S (on the right) via $\phi\colon G \to \mathrm{Perm}(S)$.

(i) The orbit of $s \in S$ is the set

$$\mathrm{orb}(s) = \{s.\phi(g) \mid g \in G\}.$$

(ii) The stabilizer of s in G is

$$\mathrm{stab}(s) = \{g \in G \mid s.\phi(g) = s\}.$$

In terms of the action graph

(i) The orbit of $s \in S$ is the connected component containing s.

(ii) The stabilizer of $s \in S$ are the group elements whose paths start and end at s; "loops."

The third local feature: fixators

Our first two local features were specific to a certain element $s \in S$.

Our last local feature is defined for each group element $g \in G$. A natural question to ask is:

(iii) What *states* (in S) does g fix?

Definition

Suppose that G acts on a set S (on the right) via $\phi\colon G \to \mathrm{Perm}(S)$.

(iii) The fixator of $g \in G$ are the elements $s \in S$ fixed by g:

$$\mathrm{fix}(g) = \big\{ s \in S \mid s.\phi(g) = s \big\}.$$

In terms of the action graph

(iii) The fixator of $g \in G$ are the nodes from which the g-paths are loops.

In terms of the "group switchboard analogy"

(i) The orbit of $s \in S$ are the elements in S that can be reached by pressing some combination of buttons.

(ii) The stabilizer of $s \in S$ consists of the buttons that have no effect on s.

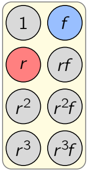
(iii) The fixator of $g \in G$ are the elements in S that don't move when we press the g-button.

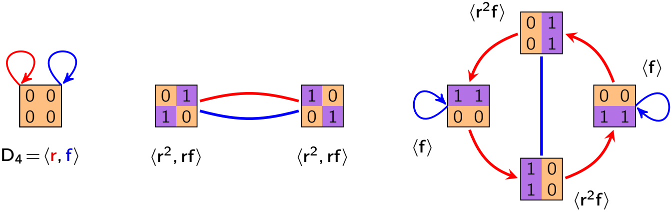
Three local features: orbits, stabilizers, and fixators

The orbits of our running example are the 3 connected components.

Each node is labeled by its stabilizer.

"Group switchboard"





The fixators are $\text{fix}(1) = S$, and

$$\text{fix}(r) = \text{fix}(r^3) = \left\{ \boxed{\begin{smallmatrix}0&0\\0&0\end{smallmatrix}} \right\} \qquad \text{fix}(r^2) = \text{fix}(rf) = \text{fix}(r^3 f) = \left\{ \boxed{\begin{smallmatrix}0&0\\0&0\end{smallmatrix}}, \boxed{\begin{smallmatrix}0&1\\1&0\end{smallmatrix}}, \boxed{\begin{smallmatrix}1&0\\0&1\end{smallmatrix}} \right\}$$

$$\text{fix}(f) = \left\{ \boxed{\begin{smallmatrix}0&0\\0&0\end{smallmatrix}}, \boxed{\begin{smallmatrix}0&0\\1&1\end{smallmatrix}}, \boxed{\begin{smallmatrix}1&1\\0&0\end{smallmatrix}} \right\} \qquad \text{fix}(r^2 f) = \left\{ \boxed{\begin{smallmatrix}0&0\\0&0\end{smallmatrix}}, \boxed{\begin{smallmatrix}0&1\\0&1\end{smallmatrix}}, \boxed{\begin{smallmatrix}1&0\\1&0\end{smallmatrix}} \right\}$$

Local duality: stabilizers vs. fixators

Consider the following table, where a checkmark at (g, s) means "*g fixes s*."

	$\begin{smallmatrix}0&0\\0&0\end{smallmatrix}$	$\begin{smallmatrix}0&1\\1&0\end{smallmatrix}$	$\begin{smallmatrix}1&0\\0&1\end{smallmatrix}$	$\begin{smallmatrix}0&0\\1&1\end{smallmatrix}$	$\begin{smallmatrix}0&1\\0&1\end{smallmatrix}$	$\begin{smallmatrix}1&1\\0&0\end{smallmatrix}$	$\begin{smallmatrix}1&0\\1&0\end{smallmatrix}$
1	✓	✓	✓	✓	✓	✓	✓
r	✓						
r^2	✓	✓	✓				
r^3	✓						
f	✓			✓		✓	
rf	✓	✓	✓				
r^2f	✓				✓		✓
r^3f	✓	✓	✓				

- the stablizers can be read off the columns: *group elements that fix $s \in S$*
- the fixators can be read off the rows: *set elements fixed by $g \in G$*.

The stabilizer subgroup

Notice how in our example, the stabilizer of each $s \in S$ is a subgroup.

This holds true for any action.

Proposition

For any $s \in S$, the set $\mathbf{stab}(s)$ is a subgroup of G.

Proof (outline)

To show $\mathbf{stab}(s)$ is a group, we need to show three things:

(i) **Identity**. That is, $s.\phi(1) = s$.

(ii) **Inverses**. That is, if $s.\phi(g) = s$, then $s.\phi(g^{-1}) = s$.

(iii) **Closure**. That is, if $s.\phi(g) = s$ and $s.\phi(h) = s$, then $s.\phi(gh) = s$.

Alternatively, it suffices to show that if $s.\phi(g) = s$ and $s.\phi(h) = s$, then $s.\phi(gh^{-1}) = s$,

You'll do this on the homework.

All three of these are very intuitive in our our switchboard analogy.

The stabilizer subgroup

As we've seen, elements in the same orbit can have different stabilizers.

Proposition (HW exercise)

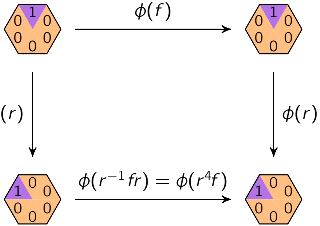
Set elements in the same orbit have conjugate stabilizers:

$$\mathrm{stab}(s.\phi(g)) = g^{-1}\,\mathrm{stab}(s)g, \quad \text{for all } g \in G \text{ and } s \in S.$$

In other words, if x stabilizes s, then $g^{-1}xg$ stabilizes $s.\phi(g)$.

Here are several ways to visualize what this means and why.

$$\begin{tikzcd}
s \arrow[r, mapsto, "\phi(x)"] \arrow[d, mapsto, "\phi(g)"'] & s \arrow[d, mapsto, "\phi(g)"] \\
s' \arrow[r, mapsto, "\phi(g^{-1}xg)"] & s'
\end{tikzcd}$$

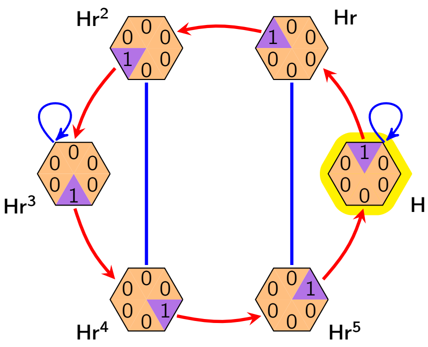


In other words, if x is a loop from s, and $s \xrightarrow{g} s'$, then $g^{-1}xg$ is a loop from s'.

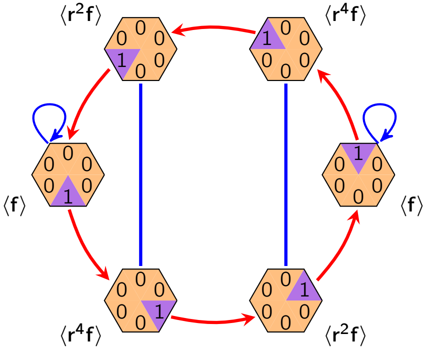
The stabilizer subgroup

Here is another example of an action (or G-set), this time of $G = D_6$.

Let s be the highlighted hexagon, and $H = \mathsf{stab}(s)$.



labeled by destinations



labeled by stabilizers

Two global features: fixed points and the kernel

Our last two features are properties of the action ϕ, rather than of specific elements.

One definition is new, and the othher is an familiar concept in this new setting.

Definition

Suppose that G acts on a set S via $\phi\colon G \to \mathsf{Perm}(S)$.

(iv) The kernel of the action is the set

$$\mathsf{Ker}(\phi) = \{k \in G \mid \phi(k) = e\} = \{k \in G \mid s.\phi(k) = s \text{ for all } s \in S\}.$$

(v) The fixed points of the action, denoted $\mathsf{Fix}(\phi)$, are the orbits of size 1:

$$\mathsf{Fix}(\phi) = \{s \in S \mid s.\phi(g) = s \text{ for all } g \in G\}.$$

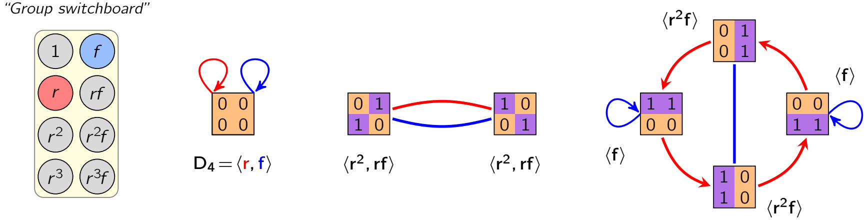
Proposition (global duality: fixed points vs. kernel)

Suppose that G acts on a set S via $\phi\colon G \to \mathsf{Perm}(S)$. Then

$$\mathsf{Ker}(\phi) = \bigcap_{s \in S} \mathsf{stab}(s), \qquad \text{and} \qquad \mathsf{Fix}(\phi) = \bigcap_{g \in G} \mathsf{fix}(g).$$

Let's also write $\mathsf{Orb}(\phi)$ for the set of orbits of ϕ.

Two global features: fixed points and the kernel



In terms of the action graph

(iv) The kernel of ϕ are the paths that are "loops from every $s \in S$."

(v) The fixed points of ϕ are the size-1 connected components.

In terms of the group switchboard analogy

(iv) The kernel of ϕ are the "broken buttons"; those $g \in G$ that have no effect on any s.

(v) The fixed points of ϕ are those $s \in S$ that are not moved by pressing any button.

Global duality: fixed points vs. kernel

Consider the following table, where a checkmark at (g, s) means g fixes s.

	$\begin{smallmatrix}0&0\\0&0\end{smallmatrix}$	$\begin{smallmatrix}0&1\\1&0\end{smallmatrix}$	$\begin{smallmatrix}1&0\\0&1\end{smallmatrix}$	$\begin{smallmatrix}0&0\\1&1\end{smallmatrix}$	$\begin{smallmatrix}0&1\\0&1\end{smallmatrix}$	$\begin{smallmatrix}1&1\\0&0\end{smallmatrix}$	$\begin{smallmatrix}1&0\\1&0\end{smallmatrix}$
1	✓	✓	✓	✓	✓	✓	✓
r	✓						
r^2	✓	✓	✓				
r^3	✓						
f	✓			✓		✓	
rf	✓	✓	✓				
r^2f	✓				✓		✓
r^3f	✓	✓	✓				

- the fixed points consist of columns with all checkmarks: *set elts fixed by everything*
- the kernel consists of the rows with all checkmarks: *group elements that fix everything.*

Two theorems on orbits, and their consequences

Our binary square example gives us some key intutition about group actions.

Qualitative observations

- elements in larger orbits tend to have smaller stabilizers, and vice-versa
- actions whose fixed point tables have more "checkmarks" tend to have more orbits.

Both of these qualitative observations can be formalized into quantitative theorems.

Theorems

1. **Orbit-stabilizer theorem**: the size of an orbit is the index of the stabilizer.
2. **Orbit-counting theorem**: the number of orbits is the average number of things fixed by a group element.

If we set up our group actions correctly, the orbit-stabilizer theorem will imply:

- The size of the conjugacy class $\mathrm{cl}_G(H)$ is the index of the normalizer of $H \leq G$
- The size of the conjugacy class $\mathrm{cl}_G(x)$ is the index of the centralizer of $x \in G$

We can also determine the number of conjugacy classes from the orbit-counting theorem.

Our first theorem on orbits

Orbit-stabilizer theorem

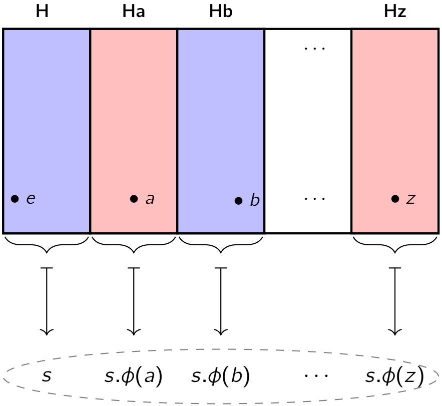
For any group action $\phi\colon G \to \mathrm{Perm}(S)$, and $s \in S$, the size of the orbit containing s is

$$|\operatorname{orb}(s)| = [G : \operatorname{stab}(s)].$$

By Lagrange's theorem, this says that $|\operatorname{orb}(s)| \cdot |\operatorname{stab}(s)| = |G|$.

Let $\mathbf{H} = \operatorname{stab}(\mathbf{s})$

applying to $s \in S$ *anything in this coset of* $\operatorname{stab}(s)$ *. . .*



$[G : \operatorname{stab}(s)]$ *cosets*

. . . yields this element in $\operatorname{orb}(s)$

$|\operatorname{orb}(s)|$ *elements*

Our first theorem on orbits

Orbit-stabilizer theorem

For any group action $\phi\colon G \to \mathrm{Perm}(S)$, and $s \in S$, the size of the orbit containing s is

$$|\,\mathrm{orb}(s)| = [G : \mathrm{stab}(s)].$$

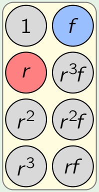
By Lagrange's theorem, this says that $|\,\mathrm{orb}(s)| \cdot |\,\mathrm{stab}(s)| = |G|$.

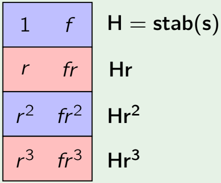
Proof

Goal: Exhibit a bijection between elements of $\mathrm{orb}(s)$, and right cosets of $\mathrm{stab}(s)$.

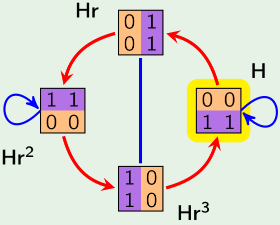
That is, "*two g-buttons send s to the same place iff they're in the same coset*".

"Group switchboard"





1	f	$\mathbf{H} = \mathbf{stab(s)}$
r	fr	$\mathbf{Hr}$
r^2	fr^2	$\mathbf{Hr^2}$
r^3	fr^3	$\mathbf{Hr^3}$



Note that $s.\phi(a) = s.\phi(b)$ iff a and b are in the same right coset of H in G.

The orbit-stabilizer theorem: $|\mathrm{orb}(s)| = [G : \mathrm{stab}(s)]$

Let $H\backslash G$ denote the set of right cosets of H in G. [Recall: G/H is the set of left cosets.]

Proof

Throughout, let $H = \mathrm{stab}(s)$. Define a map

$$f \colon H\backslash G \longrightarrow \mathrm{orb}(s), \qquad f \colon Hg \longmapsto s.\phi(g).$$

Well-defined: Suppose $Ha = Hb$. Then

$$\begin{aligned}
Hab^{-1} = H \quad &\Longrightarrow \quad ab^{-1} \in H && \text{(by the ``boring but useful coset lemma'')} \\
&\Longrightarrow \quad s.\phi(ab^{-1}) = s && \text{(by definition of stabilzer)} \\
&\Longrightarrow \quad s.\phi(a)\phi(b^{-1}) = s && \text{(properties of homomorphisms)} \\
&\Longrightarrow \quad s.\phi(a)\phi(b)^{-1} = s && \text{(properties of homomorphisms)} \\
&\Longrightarrow \quad s.\phi(a) = s.\phi(b) && \text{(right-multiply by } \phi(b)\text{)} \\
&\Longrightarrow \quad f(Ha) = f(Hb) && \text{(by definition of } f\text{)}
\end{aligned}$$

One-to-one: Change each $\Longrightarrow$ into $\Longleftrightarrow$. ✓

Onto: The preimage of $s' = s.\phi(g)$ is Hg. ✓

If we have instead, a left group action, the proof carries through but using left cosets.

Our second theorem on orbits

Orbit-counting theorem

Let a finite group G act on a set S via $\phi\colon G \to \mathsf{Perm}(S)$. Then

$$|\operatorname{Orb}(\phi)| = \frac{1}{|G|}\sum_{g\in G} |\operatorname{fix}(g)|.$$

This says that the "*average number of checkmarks per row*" is the number of orbits:

	$\begin{smallmatrix}0&0\\0&0\end{smallmatrix}$	$\begin{smallmatrix}0&1\\1&0\end{smallmatrix}$	$\begin{smallmatrix}1&0\\0&1\end{smallmatrix}$	$\begin{smallmatrix}0&0\\1&1\end{smallmatrix}$	$\begin{smallmatrix}0&1\\0&1\end{smallmatrix}$	$\begin{smallmatrix}1&1\\0&0\end{smallmatrix}$	$\begin{smallmatrix}1&0\\1&0\end{smallmatrix}$
1	✓	✓	✓	✓	✓	✓	✓
r	✓						
r^2	✓	✓	✓				
r^3	✓						
f	✓			✓		✓	
rf	✓	✓	✓				
r^2f	✓				✓		✓
r^3f	✓	✓	✓				

Orbit-counting theorem: $|\operatorname{Orb}(\phi)| = \frac{1}{|G|}\sum_{g\in G}|\operatorname{fix}(g)|$.

Proof

Let's count the number of checkmarks in the fixed point table, three ways:

$$\underbrace{\sum_{g\in G}|\operatorname{fix}(g)|}_{\text{count by rows}} = \Big|\big\{(g,s)\in G\times S \mid s.\phi(g)=s\big\}\Big| = \underbrace{\sum_{s\in S}|\operatorname{stab}(s)|}_{\text{count by columns}}\,.$$

By the orbit-stabilizer theorem, we can replace each $|\operatorname{stab}(s)|$ with $|G|/|\operatorname{orb}(s)|$:

$$\sum_{s\in S}|\operatorname{stab}(s)| = \sum_{s\in S}\frac{|G|}{|\operatorname{orb}(s)|} = |G|\sum_{s\in S}\frac{1}{|\operatorname{orb}(s)|}.$$

Let's express this sum over all disjoint orbits $S=\mathcal{O}_1\cup\cdots\cup\mathcal{O}_k$ separately:

$$|G|\sum_{s\in S}\frac{1}{|\operatorname{orb}(s)|} = |G|\sum_{\mathcal{O}\in\operatorname{Orb}(\phi)}\Bigg(\underbrace{\sum_{s\in\mathcal{O}}\frac{1}{|\operatorname{orb}(s)|}}_{=1\quad(\textit{why?})}\Bigg) = |G|\sum_{\mathcal{O}\in\operatorname{Orb}(\phi)}1 = |G|\cdot|\operatorname{Orb}(\phi)|.$$

Equating this last term with the first term gives the desired result. ☐

Groups acting on elements, subgroups, and cosets

It is frequently of interest to analyze the action of a group G on its elements, subgroups, or cosets of some fixed $H \leq G$.

Often, the orbits, stabilizers, and fixed points of these actions are familiar algebraic objects.

A number of deep theorems have a slick proof via a clever group action.

Here are common examples of group actions:

- G acts on itself by multiplication.
- G acts on itself by conjugation.
- G acts on its subgroups by conjugation.
- G acts on the cosets of a fixed subgroup $H \leq G$ by multiplication.

For each of these, we'll characterize the orbits, stabilizers, fixators, fixed points, and kernel.

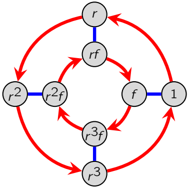
We'll encounter familiar objects such as conjugacy classes, normalizers, stabilziers, and normal subgroups, as some of our "five fundamental features".

Theorems that we have observed but haven't been able to prove yet will fall in our lap!

Groups acting on themselves by multiplication

Assume $|G| > 1$. The group G acts on itself (that is, $S = G$) by **right-multiplication**:

$$\phi\colon G \longrightarrow \mathrm{Perm}(S)\,, \qquad \phi(g) = \text{the permutation that sends each } x \mapsto xg.$$



- there is only one orbit: $\mathrm{orb}(x) = G$, for all $x \in G$
- the stabilizer of each $x \in G$ is $\mathrm{stab}(x) = \langle 1 \rangle$
- the fixator of $g \neq 1$ is $\mathrm{fix}(g) = \emptyset$.
- there are no fixed points, and the kernel is trivial:

$$\mathrm{Fix}(\phi) = \bigcap_{g \in G} \mathrm{fix}(g) = \emptyset, \qquad \text{and} \qquad \mathrm{Ker}(\phi) = \bigcap_{s \in S} \mathrm{stab}(s) = \langle 1 \rangle.$$

Cayley's theorem

If $|G| = n$, then there is an embedding $G \hookrightarrow S_n$.

Proof

Let G act on itself by right multiplication. This defines a homomorphism

$$\phi\colon G \longrightarrow \mathrm{Perm}(S) \cong S_n.$$

Since $\mathrm{Ker}(\phi) = \langle 1 \rangle$, it is an embedding. ☐

Groups acting on themselves by conjugation

Another way a group G can act on itself (that is, $S=G$) is by **right-conjugation**:

$$\phi\colon G \longrightarrow \mathrm{Perm}(S)\,, \qquad \phi(g) = \text{the permutation that sends each } x \mapsto g^{-1}xg.$$

- The orbit of $x \in G$ is its conjugacy class:

$$\mathrm{orb}(x) = \big\{x.\phi(g) \mid g \in G\big\} = \big\{g^{-1}xg \mid g \in G\big\} = \mathrm{cl}_G(x).$$

- The stabilizer of x is its centralizer:

$$\mathrm{stab}(x) = \big\{g \in G \mid g^{-1}xg = x\big\} = \big\{g \in G \mid xg = gx\big\} := C_G(x)$$

- The fixator of $g \in G$ is also its centralizer, because

$$\mathrm{fix}(g) = \big\{x \in S \mid x.\phi(g) = x\big\} = \big\{x \in G \mid g^{-1}xg = x\big\} = C_G(g).$$

- The fixed points and kernel are the center, because

$$\mathrm{Fix}(\phi) = \bigcap_{g\in G} \mathrm{fix}(g) = \bigcap_{g\in G} C_G(g) = Z(G) = \bigcap_{x\in G} C_G(x) = \bigcap_{x\in G} \mathrm{stab}(x) = \mathrm{Ker}(\phi).$$

Groups acting on themselves by conjugation

Let's apply our two theorems:

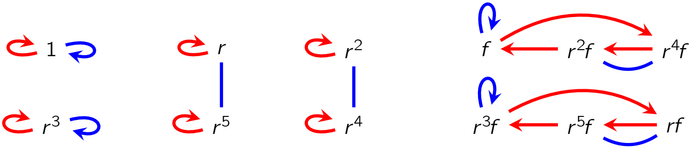
1. **Orbit-stabilizer theorem**. "*the size of an orbit is the index of the stabilizer*":

$$\left|\operatorname{cl}_G(x)\right| = [G : C_G(x)] = \frac{|G|}{|C_G(x)|}.$$

2. **Orbit-counting theorem**. "*the number of orbits is the average number of elements fixed by a group element*":

$$\#\text{conjugacy classes of } G = \text{average size of a centralizer.}$$

Let's revisit our old example of conjugacy classes in $D_6 = \langle r, f \rangle$:



Notice that the stabilizers are $\mathbf{stab}(r) = \mathbf{stab}(r^2) = \mathbf{stab}(r^4) = \mathbf{stab}(r^5) = \langle r \rangle$,

$$\mathbf{stab}(1) = \mathbf{stab}(r^3) = D_6, \qquad \mathbf{stab}(r^i f) = \langle r^3, r^i f \rangle.$$

Groups acting on themselves by conjugation

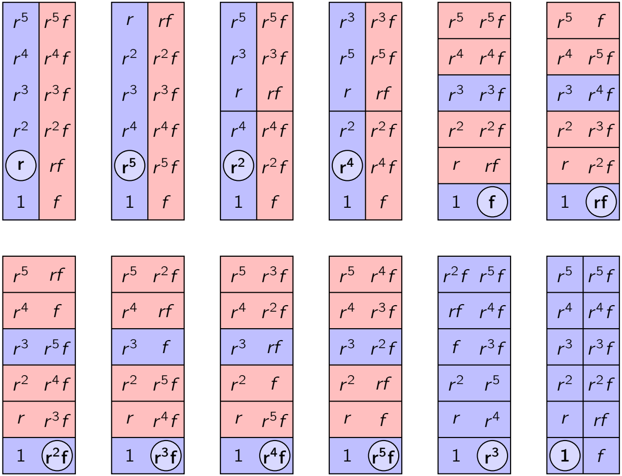
Here is the "*fixed point table*". Note that $\text{Ker}(\phi) = \text{Fix}(\phi) = \langle r^3 \rangle$.

	1	r	r^2	r^3	r^4	r^5	f	rf	r^2f	r^3f	r^4f	r^5f
1	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
r	✓	✓	✓	✓	✓	✓						
r^2	✓	✓	✓	✓	✓	✓						
r^3	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
r^4	✓	✓	✓	✓	✓	✓						
r^5	✓	✓	✓	✓	✓	✓						
f	✓			✓			✓			✓		
rf	✓			✓				✓			✓	
r^2f	✓			✓					✓			✓
r^3f	✓			✓			✓			✓		
r^4f	✓			✓				✓			✓	
r^5f	✓			✓					✓			✓

By the **orbit-counting theorem**, there are $|\,\text{Orb}(\phi)| = 72/|D_6| = 6$ conjugacy classes.

Groups acting on themselves by conjugation

Here are the cosets of all 12 cyclic subgroups in D_6 (some coincide).



Do you see how to deduce from the orbit-counting theorem that there are 6 conjugacy classes?

Groups acting on subgroups by conjugation

Any group G acts on its set S of subgroups by **right-conjugation**:

$$\phi\colon G \longrightarrow \mathrm{Perm}(S)\,, \qquad \phi(g) = \text{the permutation that sends each } H \text{ to } g^{-1}Hg.$$

This is a **right action**, but there is an associated left action: $H \mapsto gHg^{-1}$.

Let $H \leq G$ be an element of S.

- The orbit of H consists of all conjugate subgroups:

$$\mathrm{orb}(H) = \{g^{-1}Hg \mid g \in G\} = \mathrm{cl}_G(H).$$

- The stabilizer of H is the normalizer of H in G:

$$\mathrm{stab}(H) = \{g \in G \mid g^{-1}Hg = H\} = N_G(H).$$

- The fixator of g are the subgroups that g normalizes:

$$\mathrm{fix}(g) = \{H \mid g^{-1}Hg = H\} = \{H \mid g \in N_G(H)\},$$

- The fixed points of ϕ are precisely the normal subgroups of G:

$$\mathrm{Fix}(\phi) = \{H \leq G \mid g^{-1}Hg = H \text{ for all } g \in G\}.$$

- The kernel of this action is the set of elements that normalize every subgroup:

$$\mathrm{Ker}(\phi) = \{g \in G \mid g^{-1}Hg = H \text{ for all } H \leq G\} = \bigcap_{H \leq G} N_G(H).$$

Groups acting on subgroups by conjugation

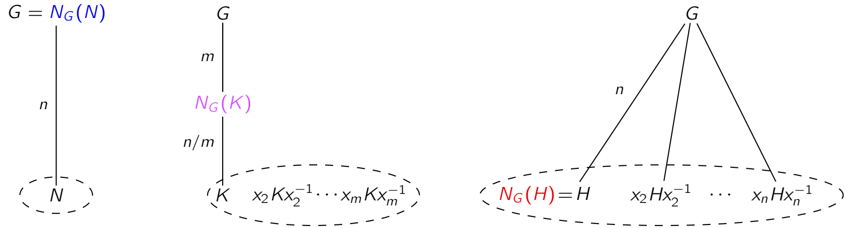
Let's apply our two theorems:

1. **Orbit-stabilizer theorem**. "*the size of an orbit is the index of the stabilizer*":

$$\big|\operatorname{cl}_G(H)\big| = [G : N_G(H)] = \frac{|G|}{|N_G(H)|}.$$

2. **Orbit-counting theorem**. "*the number of orbits is the average number of elements fixed by a group element*":

$$\#\text{conjugacy classes of subgroups of } G = \mathbb{E}\big[\#\text{ subgroups } g \text{ normalizes}\big].$$



normal

$\big|\operatorname{cl}_G(N)\big| = 1$

moderately unnormal

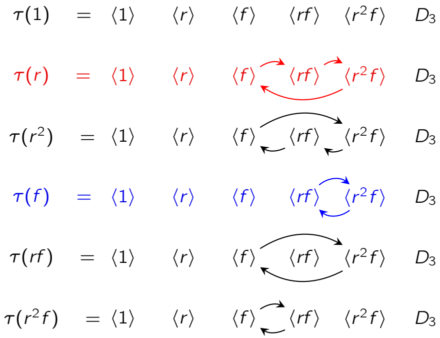
$1 < \big|\operatorname{cl}_G(K)\big| < [G : K]$

fully unnormal

$\big|\operatorname{cl}_G(H)\big| = [G : H]$; as large as possible

Groups acting on subgroups by conjugation

Here is an example of $G = D_3$ acting on its subgroups.



$\tau(1) = \langle 1\rangle \quad \langle r\rangle \quad \langle f\rangle \quad \langle rf\rangle \quad \langle r^2f\rangle \quad D_3$

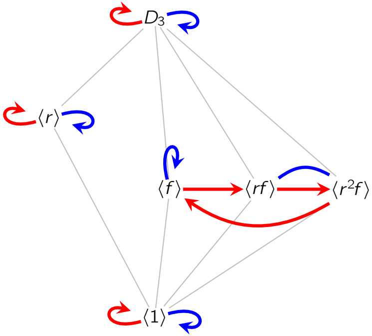
$\tau(r) = \langle 1\rangle \quad \langle r\rangle \quad \langle f\rangle \quad \langle rf\rangle \quad \langle r^2f\rangle \quad D_3$

$\tau(r^2) = \langle 1\rangle \quad \langle r\rangle \quad \langle f\rangle \quad \langle rf\rangle \quad \langle r^2f\rangle \quad D_3$

$\tau(f) = \langle 1\rangle \quad \langle r\rangle \quad \langle f\rangle \quad \langle rf\rangle \quad \langle r^2f\rangle \quad D_3$

$\tau(rf) = \langle 1\rangle \quad \langle r\rangle \quad \langle f\rangle \quad \langle rf\rangle \quad \langle r^2f\rangle \quad D_3$

$\tau(r^2f) = \langle 1\rangle \quad \langle r\rangle \quad \langle f\rangle \quad \langle rf\rangle \quad \langle r^2f\rangle \quad D_3$



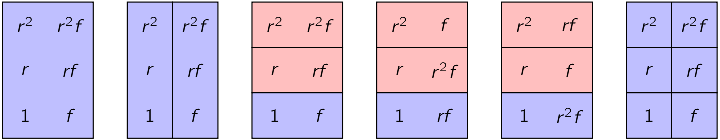
Observations

Do you see how to read stabilizers and fixed points off of the permutation diagram?

- $\text{Ker}(\phi) = \langle 1\rangle$ consists of the row(s) with only fixed points.
- $\text{Fix}(\phi) = \{\langle 1\rangle, \langle r\rangle, D_3\}$ consists of the column(s) with only fixed points.
- By the orbit-counting theorem, there are $|\text{Orb}(\phi)| = 24/|D_3| = 4$ conjugacy classes.

Groups acting on subgroups by conjugation

Consider the partitions of D_3 by the left cosets of its six subgroups:



D_3/D_3 $\quad D_3/\langle r\rangle$ $\quad D_3/\langle f\rangle$ $\quad D_3/\langle rf\rangle$ $\quad D_3/\langle r^2f\rangle$ $\quad D_3/\langle 1\rangle$

- fix(g) are the subgroups H for which "*g appears in a blue coset of H*"
- Ker(ϕ) are elements that "*only appear in blue cosets*"
- By the orbit-counting theorem, the subgroups fall into

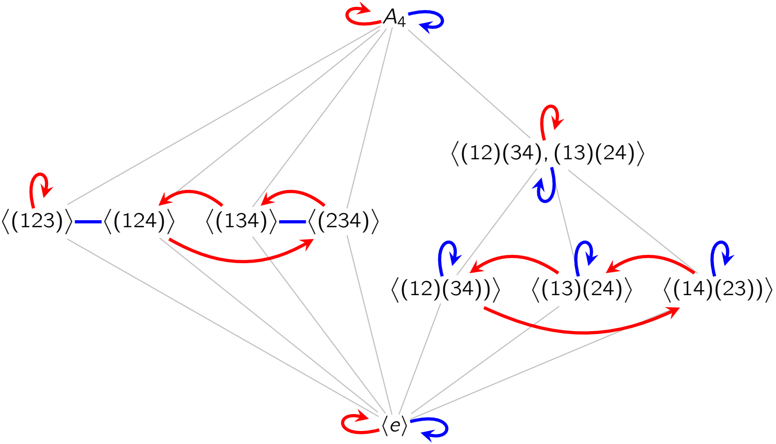
$$|\operatorname{Orb}(\phi)| = \text{average \# checkmarks per row} = \frac{\text{total \# of blue entries}}{|G|}$$

conjugacy classes.

Equivalently: *how many full "G-boxes" the blue cosets can be rearranged to fill up.*

Groups acting on subgroups by conjugation

Here is an example of $G = A_4 = \langle (123), (12)(34) \rangle$ acting on its subgroups.



Let's take a moment to revisit our "*three favorite examples*" from Chapter 3.

$$N = \langle (12)(34), (13)(24) \rangle, \qquad H = \langle (123) \rangle, \qquad K = \langle (12)(34) \rangle.$$

Groups acting on subgroups by conjugation

Here is the "*fixed point table*" of the action of A_4 on its subgroups.

	$\langle e\rangle$	$\langle(123)\rangle$	$\langle(124)\rangle$	$\langle(134)\rangle$	$\langle(234)\rangle$	$\langle(12)(34)\rangle$	$\langle(13)(24)\rangle$	$\langle(14)(23)\rangle$	$\langle(12)(34),(13)(24)\rangle$	A_4
e	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
(123)	✓	✓							✓	✓
(132)	✓	✓							✓	✓
(124)	✓		✓						✓	✓
(142)	✓		✓						✓	✓
(134)	✓			✓					✓	✓
(143)	✓			✓					✓	✓
(234)	✓				✓				✓	✓
(243)	✓				✓				✓	✓
$(12)(34)$	✓					✓	✓	✓	✓	✓
$(13)(24)$	✓					✓	✓	✓	✓	✓
$(14)(23)$	✓					✓	✓	✓	✓	✓

By the **orbit-counting theorem**, there are $|\operatorname{Orb}(\phi)| = 60/|A_4| = 5$ conjugacy classes.

Groups acting on cosets of H by multiplication

Fix a subgroup $H \leq G$. Then G acts on its **right cosets** by **right-multiplication**:

$$\phi \colon G \longrightarrow \mathrm{Perm}(S)\,, \qquad \phi(g) = \text{the permutation that sends each } Hx \text{ to } Hxg.$$

Let Hx be an element of $S = H\backslash G$ (the right cosets of H).

- There is only one orbit. For example, given two cosets Hx and Hy,

$$\phi(x^{-1}y) \;\text{ sends }\; Hx \longmapsto Hx(x^{-1}y) = Hy.$$

- The stabilizer of Hx is the conjugate subgroup $x^{-1}Hx$:

$$\mathrm{stab}(Hx) = \big\{g \in G \mid Hxg = Hx\big\} = \big\{g \in G \mid Hxgx^{-1} = H\big\} = x^{-1}Hx.$$

- There doesn't seem to be a standard term for the fixator of g:

$$\mathrm{fix}(g) = \big\{Hx \mid Hxg = Hx\big\} = \big\{Hx \mid xgx^{-1} \in H\big\}.$$

- Assuming $H \neq G$, there are no fixed points of ϕ.
- The kernel of this action is the intersection of all conjugate subgroups of H:

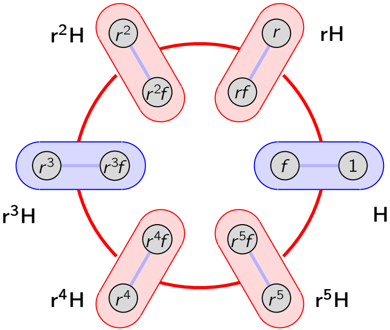
$$\mathrm{Ker}(\phi) = \bigcap_{x \in G} \mathrm{stab}(x) = \bigcap_{x \in G} x^{-1}Hx.$$

Notice that $\langle 1 \rangle \leq \mathrm{Ker}\,\phi \leq H$, and $\mathrm{Ker}(\phi) = H$ iff $H \trianglelefteq G$.

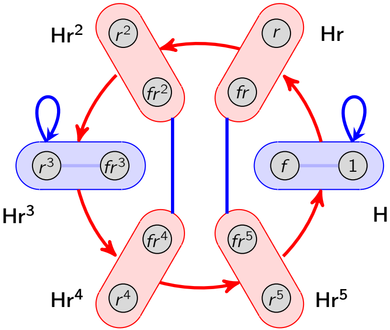
Groups acting on cosets of H by multiplication

The quotient process is done by collapsing the Cayley graph by the left cosets of H.

In contrast, this action is the result of collapsing the Cayley graph by the right cosets.



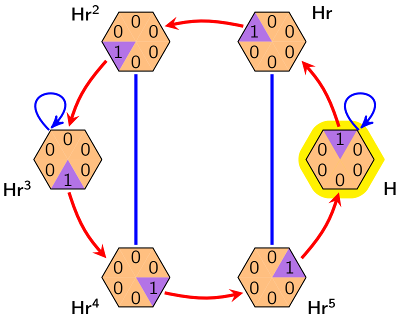
not a valid action graph

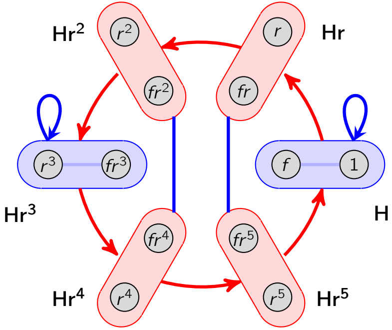


action graph of ϕ

Groups acting on cosets of H by multiplication

Soon, we'll see that *every* transitive action is equivalent to G acting on cosets of a subgroup.





This is why it's helpful to have a notion of G-set isomorphism.

In other words, we can *always* quotient by a subgroup $H \leq G$ to get a G-set.

This G-set is a group if and only if H is normal.

A summary of our four actions

Thus far, we have seen four important (right) actions of a group G, acting on:

- itself by multiplication
- itself by conjugation
- its subgroups by conjugation
- cosets of $H \leq G$ by multiplication.

set $S =$	G		subgroups of G	right cosets of H
operation	multiplication	conjugation	conjugation	right multiplication
$\text{orb}(s)$	G	$\text{cl}_G(g)$	$\text{cl}_G(H)$	$H\backslash G$
$\lvert \text{orb}(s) \rvert$	$\lvert G \rvert$	$[G : C_G(g)]$	$[G : N_G(H)]$	$[G : H]$
$\lvert \text{Orb}(\phi) \rvert$	1	avg. $\lvert \text{cl}_G(g) \rvert$	avg. $\lvert \text{cl}_G(H) \rvert$	1
$\text{stab}(s)$	$\langle 1 \rangle$	$C_G(g)$	$N_G(H)$	$x^{-1}Hx$
$\text{fix}(g)$	$\{1\}$ or $\emptyset$	$C_G(g)$	$\{H \mid g \in N_G(H)\}$	$\{Hx \mid xgx^{-1} \in H\}$
$\text{Fix}(\phi)$	none	$Z(G)$	normal subgroups	none
$\text{Ker}(\phi)$	$\langle 1 \rangle$	$Z(G)$	$\displaystyle\bigcap_{H \leq G} N_G(H)$	$\displaystyle\bigcap_{x \leq G} x^{-1}Hx$

Actions of automorphism groups

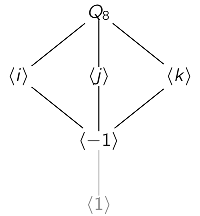
For any G, the automorphism group $\mathrm{Aut}(G)$ naturally acts on $S = G$ via a homomorphism

$$\phi\colon \mathrm{Aut}(G) \longrightarrow \mathrm{Perm}(S), \qquad \phi(\sigma) = \textit{the permutation that sends each } g \mapsto \sigma(g).$$

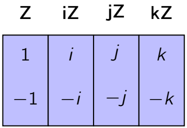
Let's see an example. Any $\sigma \in \mathrm{Aut}(Q_8)$ must send i to an element of order 4: $\pm i$, $\pm j$, $\pm k$.

This leaves 4 choices for $\sigma(j)$. Therefore, $|\,\mathrm{Aut}(Q_8)| \leq 24$.

The inner automorphism group is $N := \mathrm{Inn}(Q_8) = \{\, \mathrm{Id}, \varphi_i, \varphi_j, \varphi_k \,\}$.

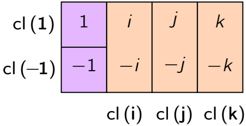


$\mathrm{Inn}(Q_8) \cong Q_8/\langle -1 \rangle \cong V_4$



Z	iZ	jZ	kZ
1	i	j	k
-1	$-i$	$-j$	$-k$

cosets of $Z(Q_8)$ are in bijection with inner automorphisms of Q_8



cl (1)	1	i	j	k
cl (−1)	-1	$-i$	$-j$	$-k$
		cl (i)	cl (j)	cl (k)

inner automorphisms of Q_8 permute elements within conjugacy classes

All 6 permutations of $\{i, j, k\}$ define a subgroup $H \leq \mathrm{Aut}(Q_8)$. Since $N \cap H = \langle \mathrm{Id} \rangle$,

$$\mathrm{Aut}(Q_8) \cong \mathrm{Inn}(Q_8) \rtimes \underbrace{H}_{\cong S_3} = \mathrm{Inn}(Q_8) \rtimes \mathrm{Out}(Q_8) \cong V_4 \rtimes S_3 \cong S_4.$$

Automorphisms of Q_8

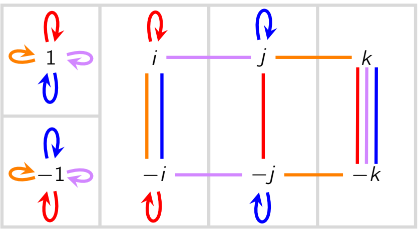
The group $\mathrm{Aut}(Q_8)$ naturally acts on the set S of...

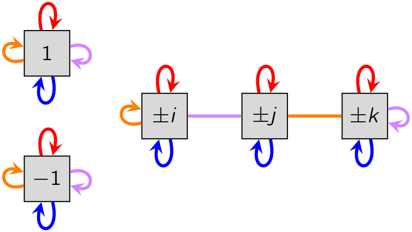
- elements of Q_8, via

$$\phi\colon \mathrm{Aut}(G) \longrightarrow \mathrm{Perm}(S), \qquad \phi(\sigma) = \textit{the permutation that sends each } g \mapsto \sigma(g).$$

- conjugacy classes of Q_8, via

$$\theta\colon \mathrm{Aut}(G) \longrightarrow \mathrm{Perm}(S), \qquad \theta(\sigma) = \textit{the permutation sending each } \mathrm{cl}_G(g) \mapsto \mathrm{cl}_G(\sigma(g)).$$

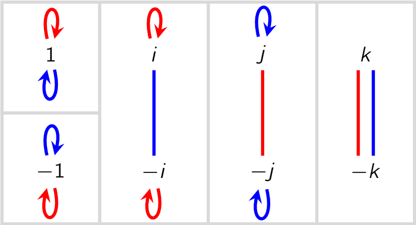




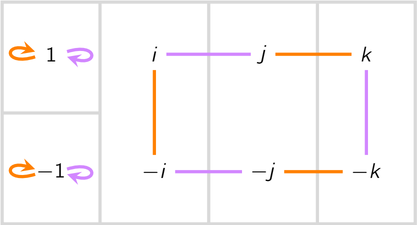
Automorphisms of Q_8

There are also actions by the inner and outer automorphism groups.

$\mathrm{Inn}(Q_8) \cong V_4$ *acting on* $S = Q_8$.



$\mathrm{Out}(Q_8) \cong S_3$ *does not act on* $S = Q_8$



These groups can also act on the:

- conjugacy classes of G,
- set of subgroups of G.

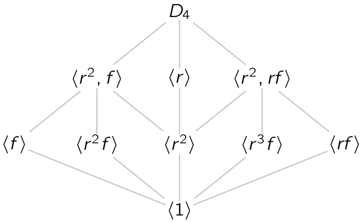
Characteristic subgroups

Definition

A subgroup $H \leq G$ is characteristic, written $H \operatorname{\mathbf{char}} G$ or $H \blacktriangleleft G$, if $\sigma(H) = H$ for all $\sigma \in \operatorname{Aut}(G)$.

Examples of characteristic subgroups are the center $Z(G)$ and commutator subgroup G'.

Normality is *not* transitive: $K \trianglelefteq H \trianglelefteq G$ does not imply $K \trianglelefteq G$.

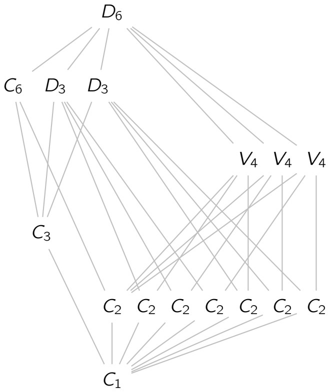


Proposition

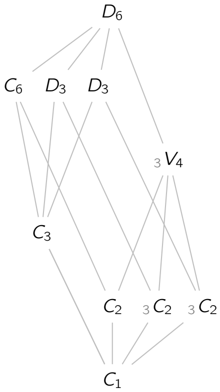
Being characteristic *is* transitive: $K \blacktriangleleft H \blacktriangleleft G$ implies $K \blacktriangleleft G$.

Characterstic subgroup diagrams

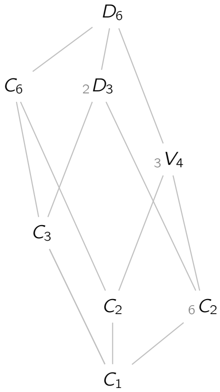
Sometimes, it is helpful to see a subgroup diagram variant, where the nodes are automorphs, instead of conjugacy classes.



subgroup lattice



subgroup diagram



automorph diagram

Other characteristic subgroups

A maximal subgroup of G is some $M \leq G$ for which $M \leq H \leq G$ implies $H = M$ or $H = G$.

Definition

The Frattini subgroup, denoted $\Phi(G)$, is the intersection of all maximal subgroups of G.

Properties

- $\Phi(G)$ is characteristic, and hence normal.
- $\Phi(G)$ is the set of non-generating elements of G:

$$\Phi(G) = \big\{a \in G \mid \text{if } a \in S \text{ and } G = \langle S \rangle, \text{ then } G = \langle S \backslash \{a\} \rangle \big\}.$$

- If H and K are finite, then $\Phi(H \times K) = \Phi(H) \times \Phi(K)$.

Definition

The scole, denoted $\mathrm{soc}(G)$, is the generated by all minimal normal subgroups of G.

If G is a finite solvable group, then $\mathrm{soc}(G)$ is a product of cyclic groups of prime order.

Action equivalence

Let's recall the difference between left-conjugating and right conjugating:

"Left group action"

$$\begin{tikzcd} H \arrow[r, "\text{conjugate by } a"] \arrow[rr, bend right, "\text{conjugate by } ba"'] & aHa^{-1} \arrow[r, "\text{conjugate by } b"] & baHa^{-1}b^{-1} \end{tikzcd}$$

$$\phi(a)\phi(b) = \phi(ba)$$

"Right group action"

$$\begin{tikzcd} H \arrow[r, "\text{conjugate by } a"] \arrow[rr, bend right, "\text{conjugate by } ab"'] & a^{-1}Ha \arrow[r, "\text{conjugate by } b"] & b^{-1}a^{-1}Hab \end{tikzcd}$$

$$\phi(a)\phi(b) = \phi(ab)$$

There's a better way to describe left actions than the faux-homomorphic $\phi(a)\phi(b) = \phi(ba)$.

"Left group action"

$$\begin{tikzcd} H \arrow[r, "\text{right-conjugate by } a^{-1}"] \arrow[rr, bend right, "\text{right-conj. by } (a^{-1}b^{-1})^{-1}"'] & aHa^{-1} \arrow[r, "\text{right-conjugate by } b^{-1}"] & baHa^{-1}b^{-1} \end{tikzcd}$$

$$\phi(a^{-1})\phi(b^{-1}) = \phi(a^{-1}b^{-1}) = \phi((ba)^{-1})$$

"Right group action"

$$\begin{tikzcd} H \arrow[r, "\text{right-conjugate by } a"] \arrow[rr, bend right, "\text{right-conj. by } ab"'] & a^{-1}Ha \arrow[r, "\text{right-conjugate by } b"] & b^{-1}a^{-1}Hab \end{tikzcd}$$

$$\phi(a)\phi(b) = \phi(ab)$$

Big idea

For every right action, there is an "equivalent" left-action where:

"pressing g-buttons, from L-to-R" $\Leftrightarrow$ "pressing g^{-1}-buttons, from R-to-L".

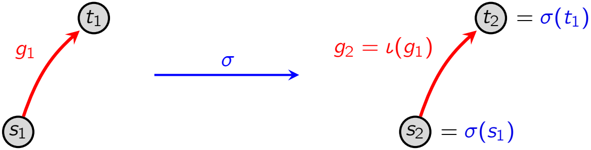
Action equivalence, informally

Action equivalence is more general. Consider two groups acting on sets, say via

$$\phi_1 \colon G_1 \longrightarrow \mathsf{Perm}(S_1), \qquad \text{and} \qquad \phi_2 \colon G_2 \longrightarrow \mathsf{Perm}(S_2).$$

If these are "equivalent", then we'll need

- a set bijection $\sigma \colon S_1 \longrightarrow S_2$
- a group isomorphism $\iota \colon G_1 \longrightarrow G_2$.



Informally, these actions are **equivalent** if:

1. pressing the g_1-button in the G_1-switchboard, followed by
2. applying $\sigma \colon S_1 \to S_2$ to get to the other graph

is the same as doing these steps in reverse order. That is,

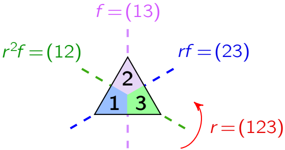
1. applying $\sigma \colon S_1 \to S_2$ to get to the other graph, then
2. pressing the $\iota(g_1)$-button on the G_2-switchboard.

A familiar example of equivalent actions

We've seen the groups:

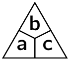
- D_3 act on a set X of six triangles,
- S_3 act on a set X' of six permuations of **123**.

These two actions are equivalent.



$X' = \{\mathbf{123}, \mathbf{132}, \mathbf{213}, \mathbf{231}, \mathbf{312}, \mathbf{321}\}$

Set bijection



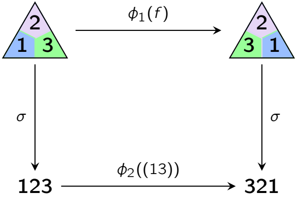
$\sigma:$ $\longmapsto$ **abc**

Isomorphism

$\iota\colon D_3 \longrightarrow S_3$

$\iota\colon r \longmapsto (123)$

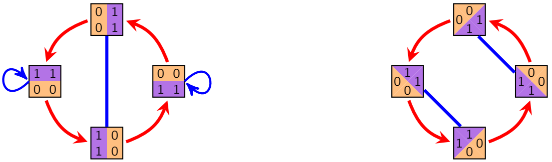
$\iota\colon f \longmapsto (23)$



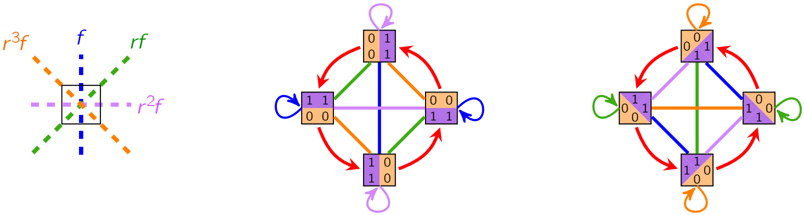
Equivalence of actions

Consider the following two sets:

Should the following two D_4-actions be considered "equivalent"?



What if we add generators?



Action equivalence, formally

Definition

Two actions $\phi_1 \colon G_1 \longrightarrow \mathsf{Perm}(S_1)$ and $\phi_2 \colon G_2 \longrightarrow \mathsf{Perm}(S_2)$ are equivalent if there is an isomorphism $\iota \colon G_1 \to G_2$ and a bijection $\sigma \colon S_1 \to S_2$ such that

$$\sigma \circ \phi_1(g) = \phi_2(\iota(g)) \circ \sigma, \quad \text{for all } g \in G.$$

We say that the resulting action graphs are action equivalent.

If $G_1 = G_2$ and $\iota \colon G \to G$ is the identity map, then S_1 and S_2 are isomorphic as G-sets.

This can be expressed with a commutative diagram:

$$\begin{tikzcd}
S_1 \arrow[r, "\phi_1(g)"] \arrow[d, "\sigma"'] & S_1 \arrow[d, "\sigma"] \\
S_2 \arrow[r, "\phi_2(\iota(g))"'] & S_2
\end{tikzcd}$$

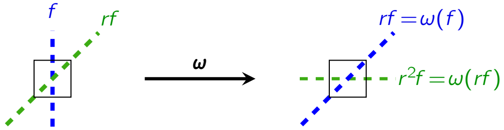
Action equivalence can be used to show that in our binary square example, we could have:

- defined $\phi(r)$ to rotate clockwise, and $\phi(f)$ to flip vertically
- used tiles with a and b, rather than 0 and 1
- read from right-to-left, rather than left-to-right, etc.

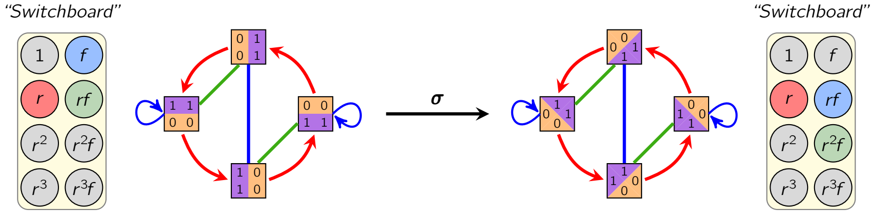
Equivalence of actions

Consider the following two sets:

The map $\sigma\colon S \longrightarrow S'$ and outer automorphism $\omega \in \mathrm{Aut}(D_4)$



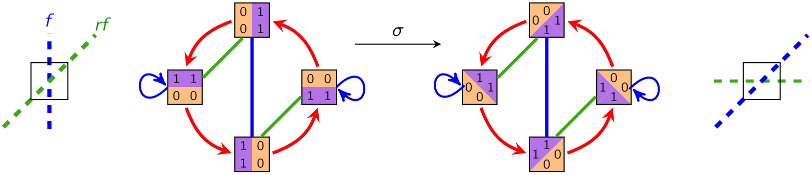
define an equivalence between the following actions:



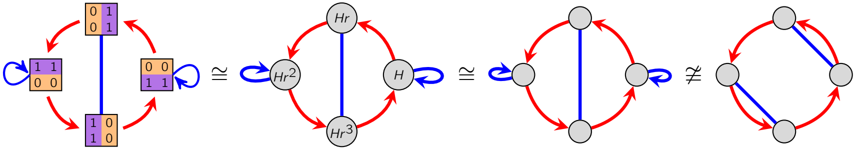
Action equivalence (weaker) vs. G-set isomorphism (stronger)

Just like we did for groups, we formalized what it means for G-sets to be isomorphic.

Since $\iota \colon G \to G$ must be the identity, the following is *not* a G-set isomorphism:



Therefore, the following equivalent actions, are as D_4-sets:



Every right action has an equivalent left action

G acting on...	right action	equivalent left action
itself by multiplication	$x \mapsto xg$	$x \mapsto g^{-1}x$
itself by conjugation	$x \mapsto g^{-1}xg$	$x \mapsto gxg^{-1}$
its subgroups by conjugation	$H \mapsto g^{-1}Hg$	$H \mapsto gHg^{-1}$
cosets by multiplication	$Hx \mapsto Hxg$	$xH \mapsto g^{-1}xH$

$$\begin{tikzcd} x \arrow[r, mapsto, "\phi_R(g)"] \arrow[d, mapsto, "\sigma"'] & xg \arrow[d, mapsto, dashed, "\text{not } \sigma"] \\ x^{-1} \arrow[r, mapsto, "\theta(g)"] & gx^{-1} \end{tikzcd}$$

$$\begin{tikzcd} x \arrow[r, mapsto, "\phi_R(g)"] \arrow[d, mapsto, "\sigma"'] & xg \arrow[d, mapsto, "\sigma"] \\ x^{-1} \arrow[r, mapsto, "\phi_L(g)"] & g^{-1}x^{-1} \end{tikzcd}$$

$$\begin{tikzcd} x \arrow[r, mapsto, "\phi_R(g)"] \arrow[d, mapsto, "\mathrm{Id}"'] & xg \arrow[d, mapsto, dashed, "\text{not } \mathrm{Id}"] \\ x \arrow[r, mapsto, "\theta(g)"] & gx \end{tikzcd}$$

🟥 $x \mapsto rx$

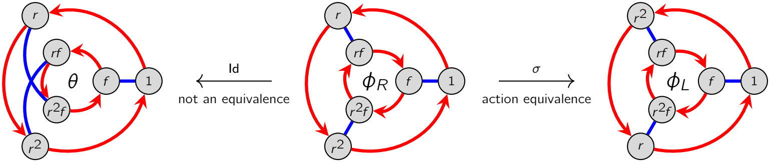
🟦 $x \mapsto fx$

🟥 $x \mapsto xr$

🟦 $x \mapsto xf$

🟥 $x \mapsto r^{-1}x = r^2x$

🟦 $x \mapsto f^{-1}x = fx$



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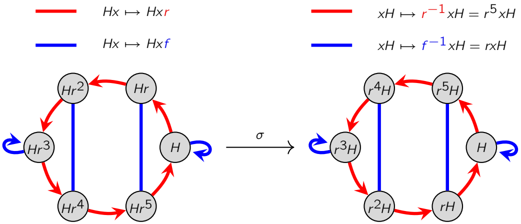
its subgroups by conjugation	$H \mapsto g^{-1}Hg$	$H \mapsto gHg^{-1}$
cosets by multiplication	$Hx \mapsto Hxg$	$xH \mapsto g^{-1}xH$

Recall that $aH = bH$ implies $Ha^{-1} = Hb^{-1}$.

$$\begin{tikzcd}
Hx \arrow[r, mapsto, "\phi_R(g)"] \arrow[d, mapsto, "\sigma"'] & Hxg \arrow[d, mapsto, "\sigma"] \\
x^{-1}H \arrow[r, mapsto, dashed, "\phi_L(g)"] & g^{-1}x^{-1}H
\end{tikzcd}$$

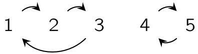


Since $aH = bH \not\Rightarrow Ha = Hb$, the the map $xH \mapsto Hx$ is not even well-defined.

Actions by permutations matrices

Consider the following permutation $\pi \in S_5$:

i	1	2	3	4	5
$\pi(i)$	2	3	1	5	4



$$\pi = (1\,2\,3)\,(4\,5)$$

The permutation matrix P_π permutes the entries of a column vector as

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_1 \\ x_2 \\ x_5 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_{\pi(1)} \\ x_{\pi(2)} \\ x_{\pi(3)} \\ x_{\pi(4)} \\ x_{\pi(5)} \end{bmatrix},$$

and the entries of a row vector as

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} x_2 & x_3 & x_1 & x_5 & x_4 \end{bmatrix}$$

$$= \begin{bmatrix} x_{\pi^{-1}(1)} & x_{\pi^{-1}(2)} & x_{\pi^{-1}(3)} & x_{\pi^{-1}(4)} & x_{\pi^{-1}(5)} \end{bmatrix}.$$

Actions by permutations matrices

In general, a left action of S_n on a set of vectors X

$$\phi_L \colon S_n \longrightarrow \mathrm{Perm}(X), \qquad \phi_L(\pi) \colon x \longmapsto P_\pi x$$

is equivalent to the right action

$$\phi_R \colon S_n \longrightarrow \mathrm{Perm}(X), \qquad \phi_R(\pi) \colon x \longmapsto x^T P_\pi^T = x^T P_{\pi^{-1}}$$

via the transpose map.

$$\begin{tikzcd}
x \arrow[r, mapsto, "\phi_L(\pi)"] \arrow[d, mapsto, "T"'] & P_\pi x \arrow[d, mapsto, "T"] \\
x^T \arrow[r, mapsto, dashed, "\phi_R(\pi)"] & x^T P_{\pi^{-1}}
\end{tikzcd}$$

$$\begin{tikzcd}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \arrow[r, mapsto, "P_\pi = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}"] \arrow[d, mapsto, "T"'] & \begin{bmatrix} x_3 \\ x_1 \\ x_2 \end{bmatrix} \arrow[d, mapsto, "T"] \\
\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \arrow[r, mapsto, dashed, "P_{\pi^{-1}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = P^T"] & \begin{bmatrix} x_3 & x_1 & x_2 \end{bmatrix}
\end{tikzcd}$$

Another equilalence between left and right actions of permutations

Recall the two "canonical" ways label a Cayley graph for $S_3 = \langle (12), (23) \rangle$ with the set

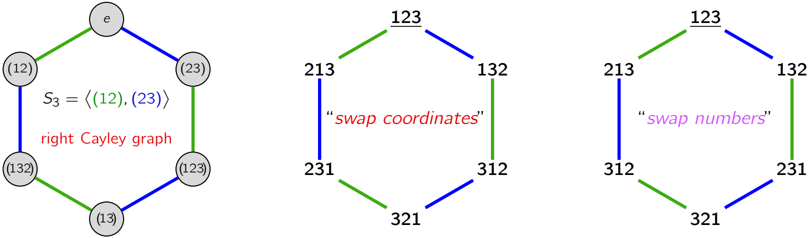
$$X = \{\mathbf{123}, \mathbf{132}, \mathbf{213}, \mathbf{231}, \mathbf{312}, \mathbf{321}\}.$$

In one, (ij) can be interpreted to mean

"*swap the numbers in the i^{th} and j^{th} coordinates*."

Alternatively, (ij) could mean

"*swap the numbers i and j, regardless of where they are*."



One of these is a right group action, and the other a left group action

Another equilalence between left and right actions of permutations

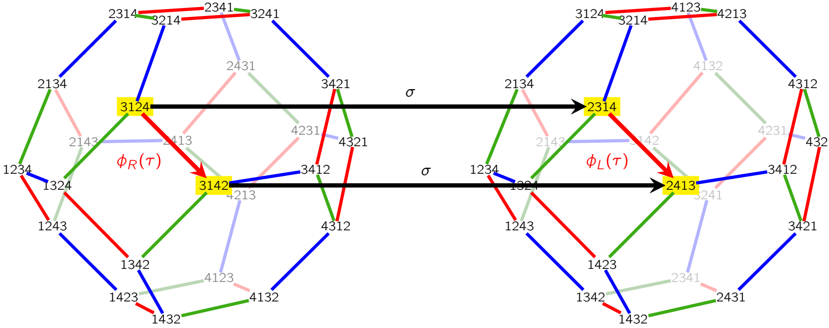
i	1	2	3	4
π	**3**	**1**	**2**	**4**

i	1	2	3	4
$\pi\tau$	**3**	**1**	**4**	**2**

$$\begin{tikzcd}
\pi(1)\pi(2)\pi(3)\pi(4) \arrow[r, mapsto, "\phi_R(\tau)", "\textit{swap coordinates 3 and 4}"'] \arrow[d, mapsto, "\sigma"'] & \pi(1)\pi(2)\pi(4)\pi(3) \arrow[d, mapsto, "\sigma"] \\
\pi^{-1}(1)\pi^{-1}(2)\pi^{-1}(3)\pi^{-1}(4) \arrow[r, mapsto, "\phi_L(\tau)", "\textit{swap digits 3 and 4}"'] & \pi^{-1}(1)\pi^{-1}(2)\pi^{-1}(4)\pi^{-1}(3)
\end{tikzcd}$$

i	1	2	3	4
π^{-1}	**2**	**3**	**1**	**4**

i	1	2	3	4
$\tau^{-1}\pi^{-1}$	**2**	**4**	**1**	**3**



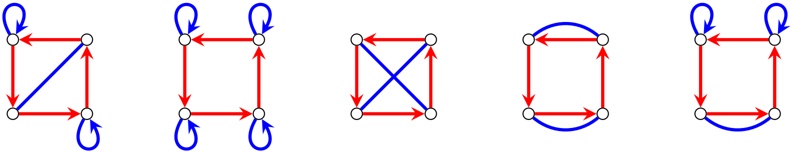
"*swap coordinates*" "*swap numbers*"

Classification of G-sets

Natural question

Given a group G, what are its possible (connected) G-sets?

For example, which of the following can arise as an orbit of an action by $G = D_4$?



Definition

An action $\phi\colon G \to \mathsf{Perm}(S)$ is

- transitive if it has only one orbit: ("*graph is connected*")
- free if $\mathsf{stab}(s) = \langle e \rangle$ for all $s \in S$. ("*uncollapsed – no nontrivial loops*")
- faithful if $\mathsf{Ker}(\phi) = \langle e \rangle$. ("*no broken buttons, except* $1 \in G$")

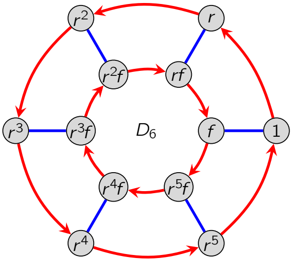
In this language our question becomes: "*classify all transitive G-actions*" (or G-sets).

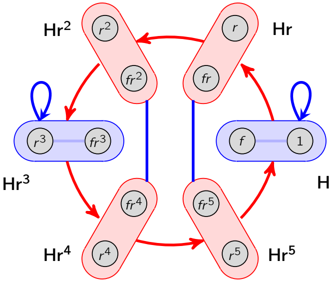
Transitive actions

Proposition

Every transitive G-action is equivalent to G acting on a set of cosets by multiplication.

A connected action graph is a Cayley graph collapsed by right cosets of some subgroup.

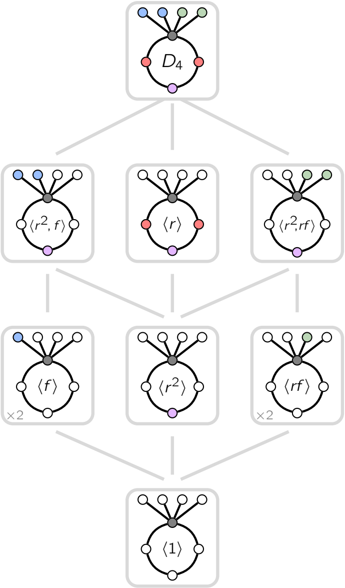


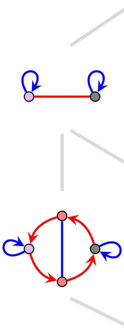


collapse right cosets of H (an action)

We can *always* collapse by right cosets. We can collapse by left cosets iff H is normal.

The transitive D_4-sets: collapsing by right cosets





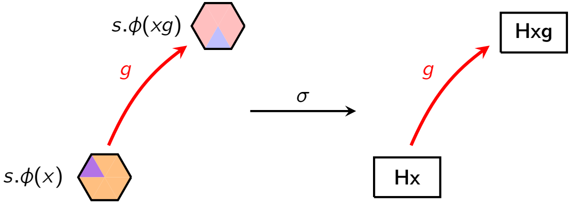
Transitive actions

Proposition

Every transitive G-action is equivalent to G acting on a set of cosets by multiplication.

Proof sketch. Let $\iota\colon G \to G$ be the identity, fix $s \in S$, let $H = \mathbf{stab}(s)$, and define

$$\sigma\colon S \longrightarrow H\backslash G, \qquad \sigma\colon s.\phi(x) \longmapsto Hx$$



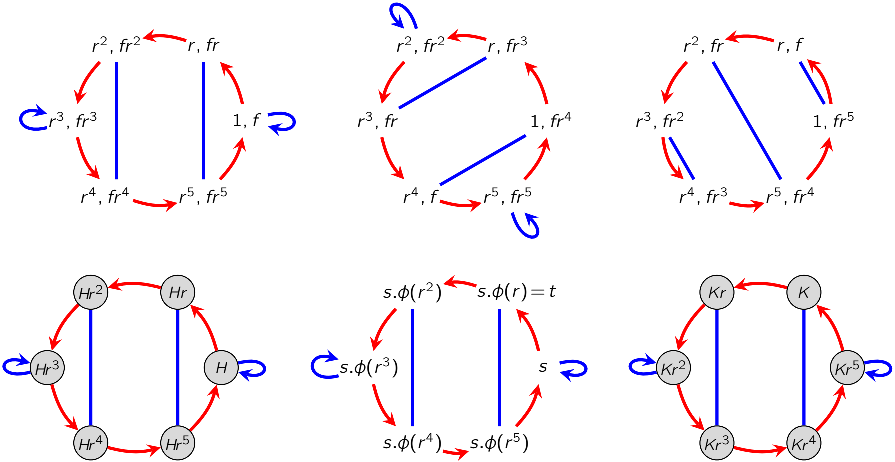
Show that σ is a well-defined bijection, and then the proof follows because:

$$\begin{tikzcd}
S \arrow[r, "\phi(g)"] \arrow[d, "\sigma"'] & S \arrow[d, "\sigma"] \\
H\backslash G \arrow[r, "\psi(g)"'] & H\backslash G
\end{tikzcd} \qquad \begin{tikzcd}
s.\phi(x) \arrow[r, "\phi(g)"] \arrow[d, "\sigma"'] & s.\phi(xg) \arrow[d, "\sigma"] \\
Hx \arrow[r, mapsto, "\psi(g)"'] & Hxg
\end{tikzcd}$$

Conjugates of $\text{stab}(s)$ give the same G-set

Proposition

If $K = a^{-1}Ha$, then $H\backslash G$ and $K\backslash G$ are isomorphic G-sets.

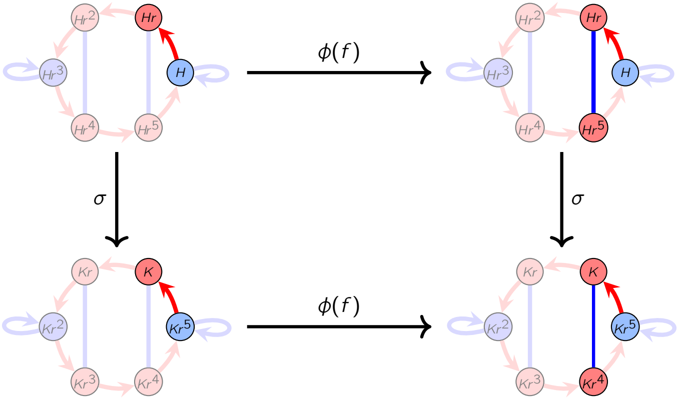


Conjugates of stab(s) give the same G-set

Proposition

If $K = a^{-1}Ha$, then $H\backslash G$ and $K\backslash G$ are isomorphic G-sets.

Consider $H = \langle f \rangle$ and $K = r^{-1}Hr = \langle r^4 f \rangle$. Define $\sigma \colon Hx \mapsto Kr^{-1}x$.



Conjugates of $\mathsf{stab}(s)$ give the same G-set

Proposition

If $K = a^{-1}Ha$, then the G-sets $H\backslash G$ and $K\backslash G$ are isomorphic.

Proof

Define the map

$$\sigma\colon H\backslash G \longrightarrow K\backslash\ G, \qquad \sigma\colon Hx \longmapsto Ka^{-1}x.$$

We claim that this is a well-defined bijection, and commutes with $\phi(g)$:

$$\begin{tikzcd}
H\backslash G \arrow[r, "\phi(g)"] \arrow[d, "\sigma"'] & H\backslash G \arrow[d, "\sigma"] \\
K\backslash G \arrow[r, "\phi(g)"'] & K\backslash G
\end{tikzcd}
\qquad
\begin{tikzcd}
Hx \arrow[r, "\phi(g)"] \arrow[d, "\sigma"'] & Hxg \arrow[d, "\sigma"] \\
Ka^{-1}x \arrow[r, mapsto, "\phi(g)"'] & Ka^{-1}xg
\end{tikzcd}$$

Well-defined: Suppose $Hx = Hy$. Then $Hyx^{-1} = H$, so $yx^{-1} \in H$.

$$\sigma(Hx) = Ka^{-1}x = \underbrace{a^{-1}H}_{=Ka^{-1}}x = a^{-1}\underbrace{(Hyx^{-1})}_{=H}x = \underbrace{a^{-1}H}_{=Ka^{-1}}y = Ka^{-1}y = \sigma(Hy).$$

Conjugates of stab(s) give the same G-set

Proposition

If $K = a^{-1}Ha$, then the G-sets $H\backslash G$ and $K\backslash G$ are isomorphic.

Proof

Define the map

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Ka^{-1}x \arrow[r, mapsto, "\phi(g)"'] & Ka^{-1}xg
\end{tikzcd}$$

Injectivity: Suppose $\sigma(Hx) = \sigma(Hy)$. Then

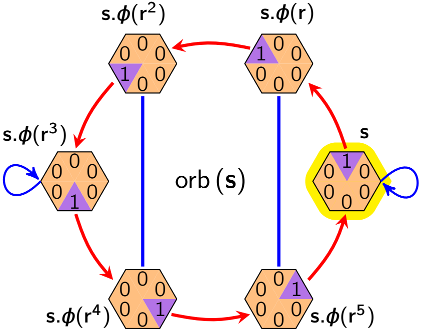
$$\sigma(Hx) = \underbrace{Ka^{-1}}_{=a^{-1}H}x = a^{-1}Hx, \qquad \text{and} \qquad \sigma(Hy) = \underbrace{Ka^{-1}}_{=a^{-1}H}y = a^{-1}Hy,$$

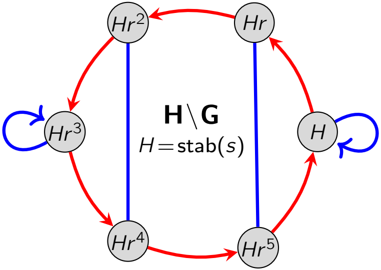
and thus $Hx = Hy$. Surjectivity is straightforward. ☐

Transitive actions

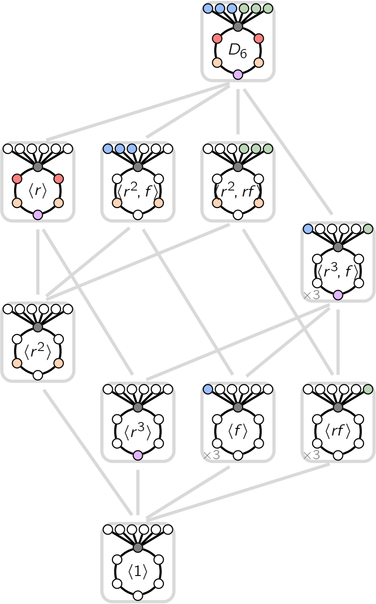
Big ideas

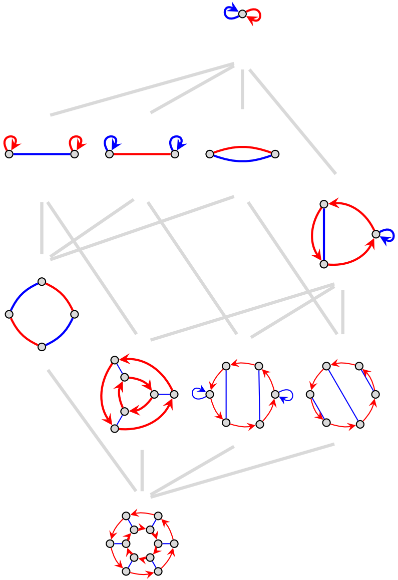
- Every transitive G-action is isomorphic to G acting on the cosets of $\textbf{stab}(s)$.
- The action graph is constructed by collapsing by right cosets of $\textbf{stab}(s)$.
- conjugates of $\textbf{stab}(s)$ give the same G-set.





The transitive D_6-sets: collapsing by right cosets





Subgroups of small index

Groups acting on cosets is a useful technique for establishing seemingly unrelated results.

Several of these involving showing that subgroups of "small index" are normal.

We've already seen that subgroups of index 2 are normal.

Of course, there are non-normal index-3 subgroups, like $\langle f \rangle \leq D_3$.

The following gives a sufficient condition for when index-3 subgroups are normal.

Proposition

If G has no subgroup of index 2, then any subgroup of index 3 is normal.

Proof

Let $H \leq G$ with $[G : H] = 3$.

Let G act on the cosets of H by multiplication, to get a nontrivial homomorphism

$$\phi \colon G \longrightarrow S_3.$$

$K := \mathrm{Ker}(\phi) \leq H$ is the largest normal subgroup of G contained in H. By the FHT,

$$G/K \cong \mathrm{Im}(\phi) \leq S_3.$$

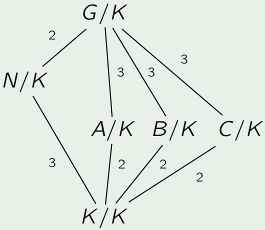
Subgroups of small index

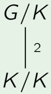
Proof (contin.)

Thus, there are three cases for this quotient:

$$G/K \cong S_3, \qquad G/K \cong C_3, \qquad G/K \cong C_2.$$

Visually, this means that we have one of the following:





By the corrdespondence theorem, $K \leq H \lneq G$ implies $K/K \leq H/K \lneq G/K$.

Since G has no index-2 subgroup, only the middle case is possible (*Why?*).

This forces $K/K = H/K$, and so $K = H$ which is normal for multiple reasons. ☐

Subgroups of small index

Proposition

Suppose $H \leq G$ and $[G : H] = p$, the smallest prime dividing $|G|$. Then $H \trianglelefteq G$.

Proof

Let G act on the cosets of H by multiplication, to get a non-trivial homomorphism

$$\phi \colon G \longrightarrow S_p.$$

The kernel $K = \mathrm{Ker}(\phi)$, is the largest normal subgroup of G such that $K \leq H \lneq G$.

We'll show that $H = K$, or equivalently, that $[H : K] = 1$. By the correspondence theorem:

$$\begin{tikzcd}
G \arrow[d, dash, "p"] \\
H \arrow[d, dash, "q \text{ is not divisible by any prime } < p"] \\
K
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
G/K \cong S_p \arrow[d, dash, "p"] \\
H/K \arrow[d, dash, "q \text{ divides } (p-1)!"] \\
K/K
\end{tikzcd}$$

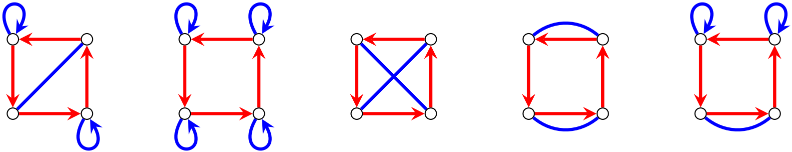
Do you see why $q = 1$? □

Classification of G-sets

Natural question

Given a group G, what are its possible (connected) G-sets?

For example, which of the following can arise as an orbit of an action by $G = D_4$?



Definition

An action $\phi \colon G \to \mathsf{Perm}(S)$ is

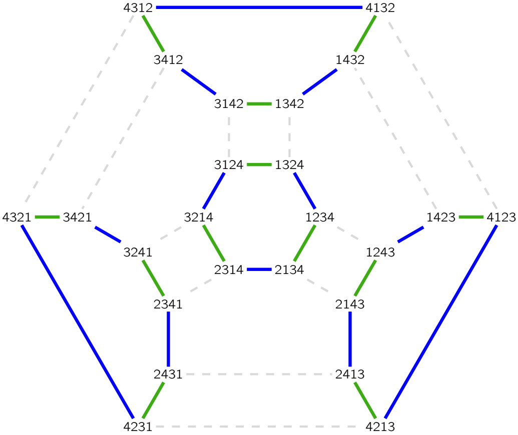
- transitive if it has only one orbit: ("*graph is connected*")
- free if $\mathsf{stab}(s) = \langle e \rangle$ for all $s \in S$. ("*uncollapsed – no nontrivial loops*")
- faithful if $\mathsf{Ker}(\phi) = \langle e \rangle$. ("*no broken buttons, except* $1 \in G$")

In this language our question becomes: "*classify all transitive G-actions*" (or G-sets).

An example of a free action that is not transitive

The group $S_3 = \langle (12), (23) \rangle$ acts on permutations **1234**, via $\phi\colon S_3 \to \mathsf{Perm}(S)$, where

- $\phi((12)) =$ the permutation that swaps the 1st and 2nd coordinates
- $\phi((23)) =$ the permutation that swaps the 2nd and 3rd coordinates

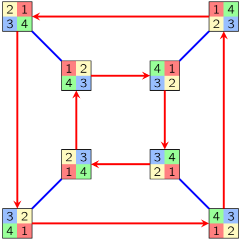


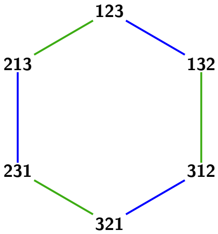
Simply transitive actions

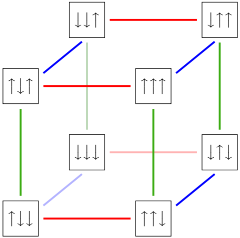
Definition

An action $\phi \colon G \to \mathrm{Perm}(S)$ is simply transitive if it is transitive and free.

Here are some simply transitive actions that we have seen.







Proposition

Every simply transitive G-action is equivalent to G acting on itself by multiplication.

This just says that *simply transitive G-sets are groups*!

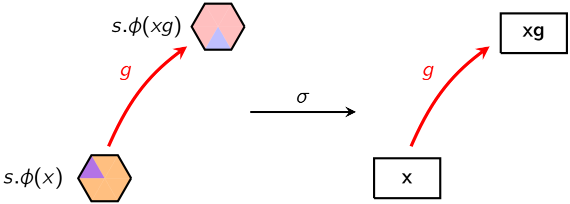
Simply transitive actions

Proposition

Every simply transitive G-action is equivalent to G acting on itself by multiplication.

Proof sketch. Let $\iota\colon G \to G$ be the identity, fix our "home state" $s \in S$, and define

$$\sigma\colon S \longrightarrow G, \qquad \sigma\colon s.\phi(x) \longmapsto x$$



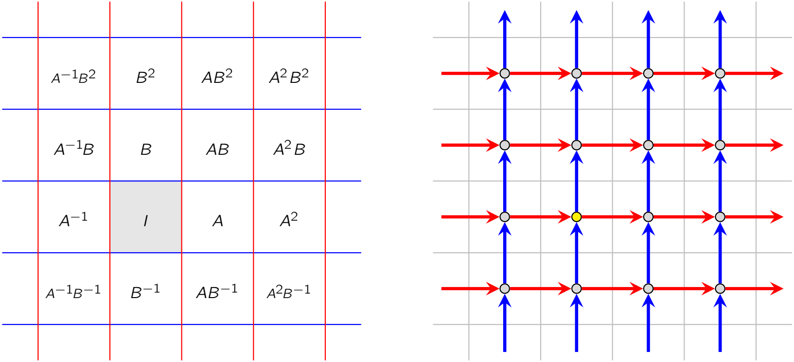
Show that σ is a well-defined bijection, and then the proof follows because:

$$\begin{tikzcd} S \arrow[r, "\phi(g)"] \arrow[d, "\sigma"'] & S \arrow[d, "\sigma"] \\ G \arrow[r, "\psi(g)"'] & G \end{tikzcd} \qquad \begin{tikzcd} s.\phi(x) \arrow[r, "\phi(g)"] \arrow[d, "\sigma"'] & s.\phi(xg) \arrow[d, "\sigma"] \\ x \arrow[r, mapsto, "\psi(g)"'] & xg \end{tikzcd}$$

Simply transitive actions from reflection groups

One place where simply transitive actions arise is from tilings.

The group $\langle A, B \mid AB = BA \rangle \cong \mathbb{Z} \times \mathbb{Z}$ acts simply transitively on the unit squares in $\mathbb{Z}^2$.

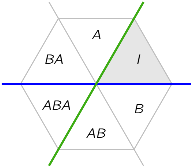


$A^{-1}B^2$	B^2	AB^2	A^2B^2
$A^{-1}B$	B	AB	A^2B
A^{-1}	I	A	A^2
$A^{-1}B^{-1}$	B^{-1}	AB^{-1}	A^2B^{-1}

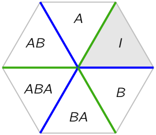
The shaded region is called a **fundamental chamber**.

Simply transitive actions from finite reflection groups

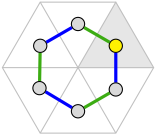
The dihedral group $D_3 = \langle A, B \mid A^2 = B^2 = (AB)^3 = 1 \rangle$ acts simply transitively on the six regions of a hexagon.



"*left action*"

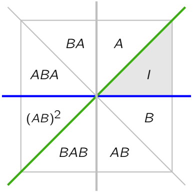


"*right action*"

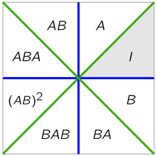


"*(right) Cayley graph*"

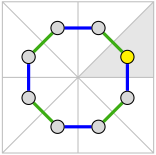
The dihedral group D_4 acts simply transitively on the eight regions of a square.



"*left action*"



"*right action*"

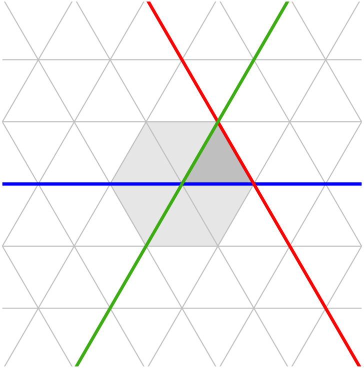


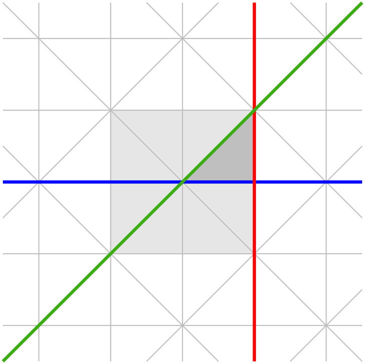
"*(right) Cayley graph*"

Simply transitive actions from affine reflection groups

In both previous examples, adding a third reflection generates a tiling of the plane.

The resulting affine groups, $\mathrm{Aff}(D_3)$ and $\mathrm{Aff}(D_4)$, act simply transitively on the chambers.

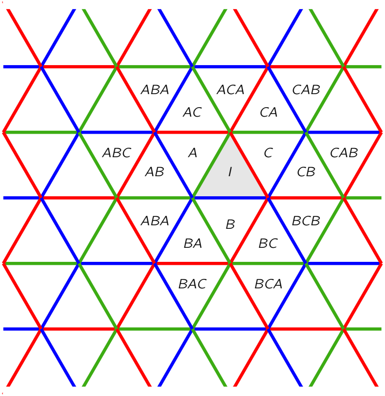


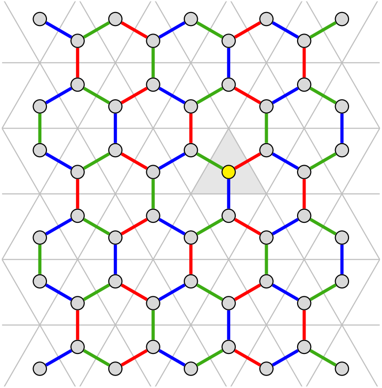


Simply transitive actions and affine Weyl groups

The group $\mathsf{Aff}(D_3)$ is better known as the affine Weyl group of type A_2.

It acts simply transitively on the chambers of the following tiling of $\mathbb{R}^2$.





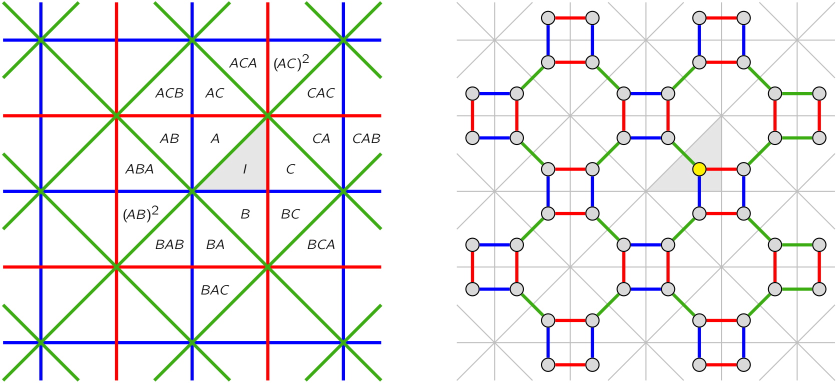
It has presentation

$$W(\tilde{A}_2) = \mathsf{Aff}(D_3) = \langle A, B, C \mid A^2 = B^2 = C^2 = (AB)^3 = (AC)^3 = (BC)^3 = 1 \rangle.$$

Simply transitive actions and affine Weyl groups

The group $\mathrm{Aff}(D_4)$ is better known as the affine Weyl group of type C_2.

It acts simply transitively on the chambers of the following tiling of $\mathbb{R}^2$.



It has presentation

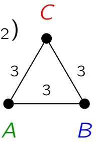
$$W(\tilde{C}_2) = \mathrm{Aff}(D_4) = \langle A, B, C \mid A^2 = B^2 = C^2 = (AB)^4 = (AC)^4 = (BC)^2 = 1 \rangle.$$

Weyl groups and Dynkin diagrams

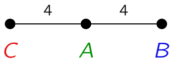
The presentations of the affine Weyl groups are encoded by Dynkin diagrams.

Nodes s_i are generators, and the labeled edges m_{ij} describe relations: $(s_i s_j)^{m_{ij}} = 1$.

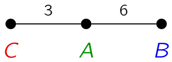
$\mathrm{Aff}(D_3) = W(\tilde{A}_2)$



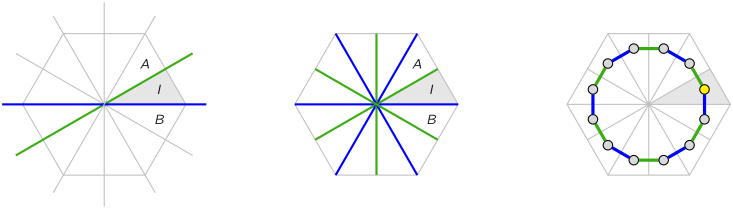
$\mathrm{Aff}(D_4) = W(\tilde{C}_2)$



$\mathrm{Aff}(D_6) = W(\tilde{G}_2)$



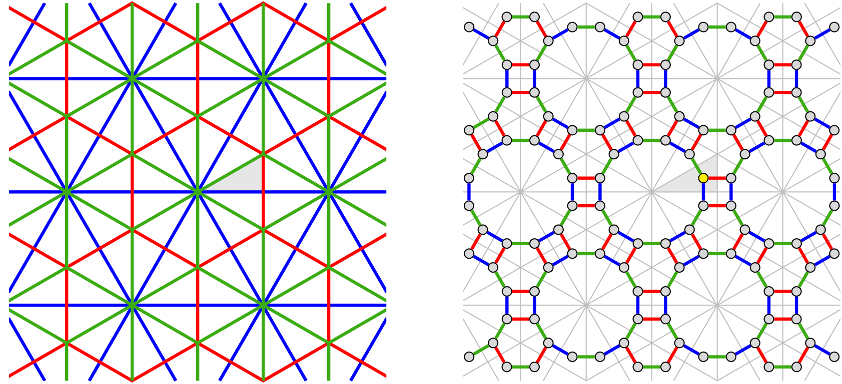
This last example is the affine version of $D_6 = \langle A, B \mid A^2 = B^2 = (AB)^6 = 1 \rangle$ acting simply transitively on the 12 regions of a hexagon.



One last affine Weyl group

The group $\text{Aff}(D_6)$ is better known as the the affine Weyl group of type G_2.

It acts simply transitively on the chambers of the following tiling of $\mathbb{R}^2$.



It has presentation

$$W(\tilde{G}_2) = \text{Aff}(D_6) = \langle A, B, C \mid A^2 = B^2 = C^2 = (AB)^6 = (AC)^3 = (BC)^2 = 1 \rangle.$$

Coxeter groups and tilings of hyperbolic space

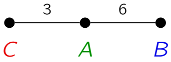
A Coxeter group is a group generated by “reflections”, with presentation

$$W = \langle s_1, \dots, s_n \mid s_i^2 = 1,\ (s_i s_j)^{m_{ij}} = 1 \rangle.$$

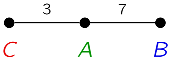
Like Weyl groups, this can be encoded by a Coxeter graph.

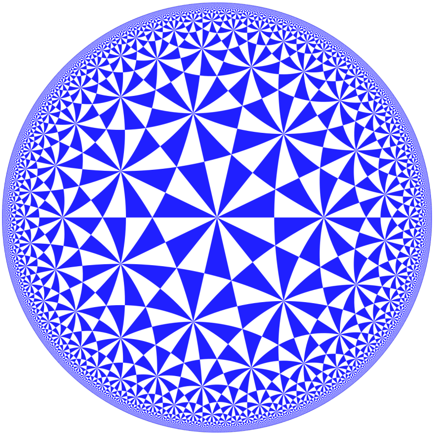
Some Coxeter groups act simply transitively on chambers of hyperbolic tilings.

$\mathrm{Aff}(D_6) = W(\tilde{G}_2)$



A hyperbolic Coxeter group





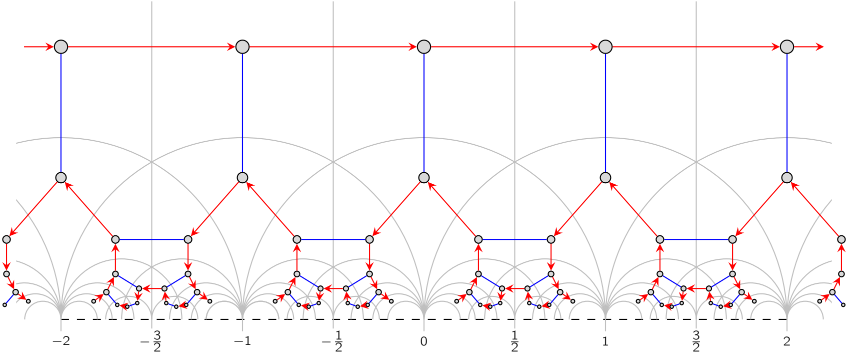
A simply transitive action of $\mathsf{PSL}_2(\mathbb{Z})$

The projective special linear group

$$\mathsf{PSL}_2(\mathbb{Z}) = \mathsf{SL}_2(\mathbb{Z})/\langle -I\rangle, \qquad \text{where } \mathsf{SL}_2(\mathbb{Z}) = \left\langle \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{S}, \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{T} \right\rangle$$

defines a tiling of hyperbolic ideal triangles in the upper half-plane via

$$S\colon z \longmapsto \frac{0z-1}{z+0} = -\frac{1}{z}, \qquad \text{and} \qquad T\colon z \longmapsto \frac{z+1}{0z+1} = z+1,$$



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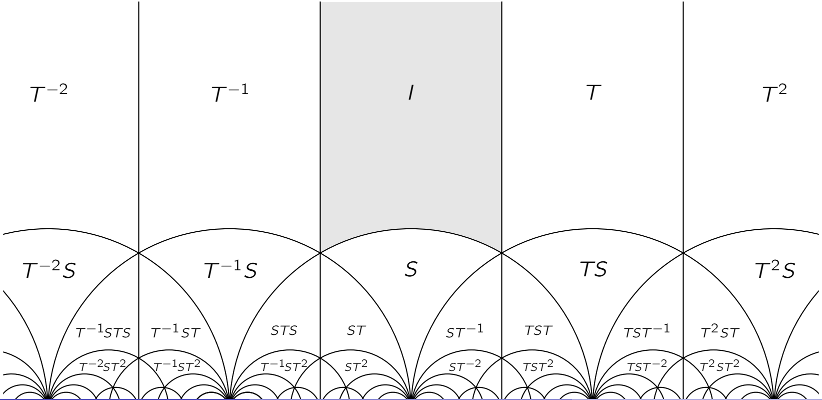
defines a tiling of hyperbolic ideal triangles in the upper half-plane via

$$S\colon z \longmapsto \frac{0z-1}{z+0} = -\frac{1}{z}, \qquad \text{and} \qquad T\colon z \longmapsto \frac{z+1}{0z+1} = z+1,$$



Equivariant maps

Definition

Suppose G acts on S_i via $\phi_i \colon G \to \mathsf{Perm}(S_i)$ for $i = 1, 2$. A **G-equivariant map** is a function $\sigma \colon S_1 \to S_2$ such that $\sigma \circ \phi_1(g) = \phi_2(g) \circ \sigma$, for all $g \in G$:

$$\begin{tikzcd} S_1 \arrow[r, mapsto, "\phi_1(g)"] \arrow[d, mapsto, "\sigma"'] & S_1 \arrow[d, mapsto, "\sigma"] \\ S_2 \arrow[r, mapsto, "\phi_2(g)"] & S_2 \end{tikzcd} \qquad \begin{tikzcd} s_1 \arrow[r, mapsto, "\phi_1(g)"] \arrow[d, mapsto, "\sigma"'] & s_1.\phi_1(g) \arrow[d, mapsto, "\sigma"] \\ s_2 \arrow[r, mapsto, "\phi_2(g)"] & s_2.\phi_2(g) \end{tikzcd}$$

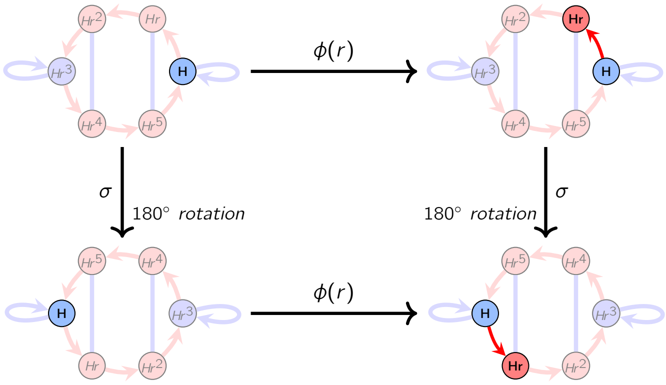
Loosely speaking, "*equivariance*" means "*commutes with the action*."

Key concepts

- Action equivalence involves an isomorphism $\iota \colon G_1 \to G_2$ and bijection $\sigma \colon S_1 \to S_2$.
- G-set isomorphisms ($\iota = \mathsf{Id}$, σ is bijective) are equivariant bijections.
- G-set automorphisms ($\iota = \mathsf{Id}$, σ is bijective, $S_1 = S_2$) form a group $\mathsf{Aut}_G(S)$.
- G-set homomorphisms ($\iota = \mathsf{Id}$, but σ need not be bijective) are equivariant maps.

G-set automorphisms as symmetries of the action graph

Let $S = G \backslash H$, for $G = D_6$ and $H = \langle f \rangle$.

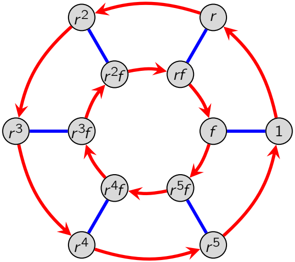


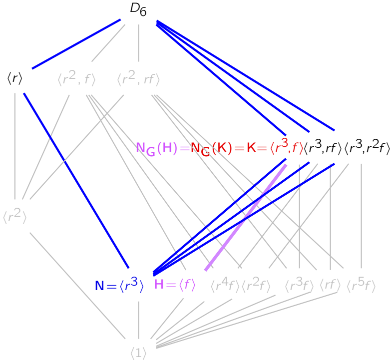
Key idea

The action and bijection clearly commute upon thinking of:

- the action $\phi(r)$ as right-multiplying Hr^i by r,
- the bijection σ as left-multiplying Hr^i by r^3. (This works because $r^3 \in N_G(H)$.)

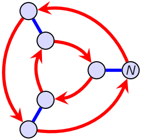
G-set automorphisms





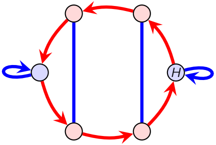
What do you notice about normalizers vs. symmetries of the actions graphs?

$N=\langle r^3\rangle$; *normal*



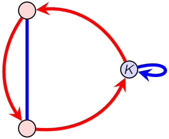
$\text{Aut}_G(N\backslash G)\cong D_3\cong N_G(N)/N$

$H=\langle f\rangle$; *moderately unnormal*



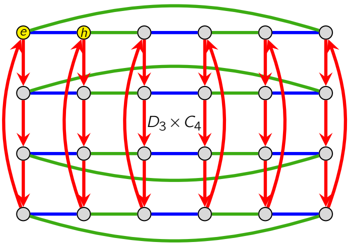
$\text{Aut}_G(H\backslash G)\cong C_2\cong N_G(H)/H$

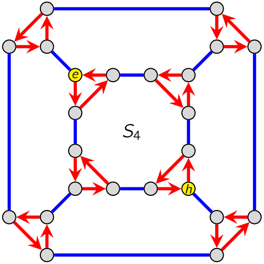
$K=\langle r^3,f\rangle$; *fully unnormal*

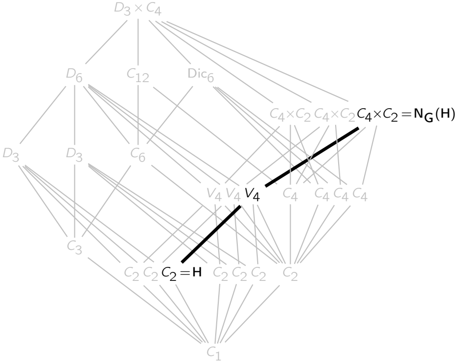


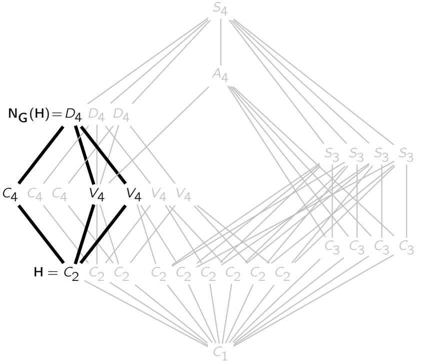
$\text{Aut}_G(K\backslash G)\cong\langle 1\rangle\cong N_G(K)/K$

G-set automorphisms

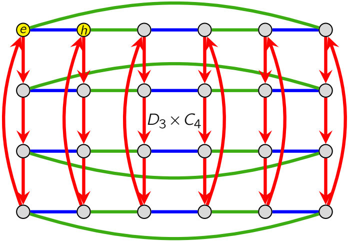






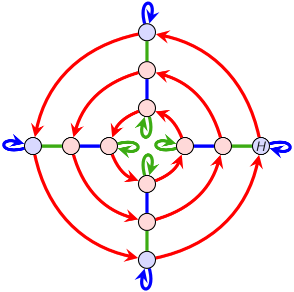


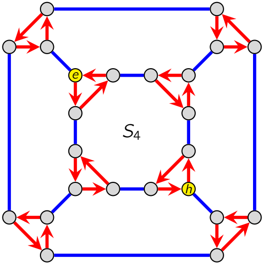
G-set automorphisms



$H = \langle (f,1) \rangle \leq D_3 \times C_4 = G$

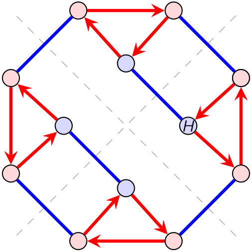
$\mathrm{Aut}_G(H \backslash G) \cong N_G(H)/H \cong C_4$





$H = \langle ((12)(34)) \rangle \leq S_4 = G$

$\mathrm{Aut}_G(H \backslash G) \cong N_G(H)/H \cong V_4$

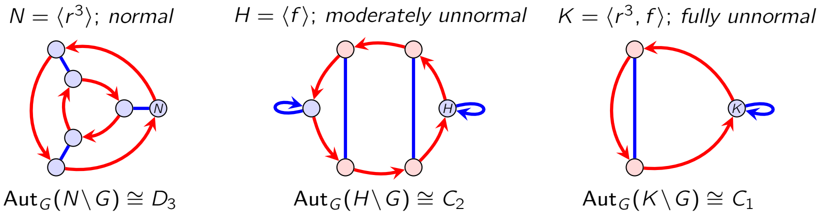


The G-set automorphism group

Theorem

For any $H \leq G$, the G-set automorphism group of $S = H\backslash G$ is

$$\mathrm{Aut}_G(S) \cong N_G(H)/H.$$


$N = \langle r^3 \rangle$; *normal*

$H = \langle f \rangle$; *moderately unnormal*

$K = \langle r^3, f \rangle$; *fully unnormal*

$\mathrm{Aut}_G(N\backslash G) \cong D_3$

$\mathrm{Aut}_G(H\backslash G) \cong C_2$

$\mathrm{Aut}_G(K\backslash G) \cong C_1$

Here's how the proof will go, given $\sigma \in \mathrm{Aut}_G(S)$, and $S = H\backslash G$:

1. **Lemma 1**: $\sigma\colon Hg \mapsto Hxg$, for some fixed $x \in G$ (call this σ_x).
2. **Lemma 2**: $\sigma_x \in \mathrm{Aut}_G(S)$ iff $x \in N_G(H)$. That is, $\sigma_x\colon Hg \mapsto xHg$.
3. **FHT**: Two $\sigma_x = \sigma_{x'}$ iff x, x' are in the same coset of H.

The G-set automorphism group

Lemma 1

Any G-set automorphism $\sigma \in \mathrm{Aut}_G(S)$, for $S = H\backslash G$, is determined by the image of H:

$$\text{if } \sigma \colon H \mapsto Hx, \quad \text{then} \quad \sigma \colon Hg \mapsto Hxg, \text{ for all } g \in G.$$

Proof

Since σ is G-equivariant, it commutes with each $\phi(g) \in \mathrm{Perm}(S)$.

That is, the following diagram commutes:

$$\begin{tikzcd}
S \arrow[r, "\phi(g)"] \arrow[d, "\sigma"'] & S \arrow[d, "\sigma"] \\
S \arrow[r, "\phi(g)"] & S
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
H \arrow[r, mapsto, "\phi(g)"] \arrow[d, mapsto, "\sigma"'] & Hg \arrow[d, mapsto, "\sigma"] \\
Hx \arrow[r, mapsto, "\phi(g)"] & Hxg
\end{tikzcd}$$

It follows that $\sigma \colon Hg \mapsto Hxg$, as claimed. $\square$

The G-set automorphism group

Lemma 2

Let $S = H\backslash G$. The map of right cosets

$$\sigma_x \colon S \longrightarrow S, \qquad \sigma_x \colon Hg \longmapsto Hxg$$

is a G-set automorphism iff $x \in N_G(H)$.

Proof

"⇒": Suppose $\sigma_x \in \mathrm{Aut}_G(H\backslash G)$, and take $h \in H$. We have:

$$\begin{tikzcd} S \arrow[r, "\phi(h)"] \arrow[d, "\sigma_x"'] & S \arrow[d, "\sigma_x"] \\ S \arrow[r, "\phi(h)"] & S \end{tikzcd} \qquad \begin{tikzcd} H \arrow[r, mapsto, "\phi(h)"] \arrow[d, mapsto, "\sigma_x"'] & H \arrow[d, mapsto, "\sigma_x"] \\ Hx \arrow[r, mapsto, "\phi(h)"] & Hxh = Hx \end{tikzcd}$$

That is, for every $h \in H$,

$$H = Hxhx^{-1} \quad \Leftrightarrow \quad xhx^{-1} \in H \quad \Leftrightarrow \quad x \in N_G(H).$$ ✓

The G-set automorphism group

Lemma 2

Let $S = H\backslash G$. The map of right cosets

$$\sigma_x \colon S \longrightarrow S, \qquad \sigma_x \colon Hg \longmapsto Hxg$$

is a G-set automorphism iff $x \in N_G(H)$.

Proof

"$\Leftarrow$": Suppose $x \in N_G(H)$, and pick $g \in G$.

By Lemma 1: $\sigma_x \colon Hg \mapsto Hxg = xHg$.

The operations σ_x (left-multiplying by x), and $\phi(g)$ (right-multiplying by g) clearly commute. ✓

$$\begin{tikzcd}
S \arrow[r, "\phi(g)"] \arrow[d, "\sigma_x"'] & S \arrow[d, "\sigma_x"] \\
S \arrow[r, "\phi(g)"] & S
\end{tikzcd}
\qquad
\begin{tikzcd}
H \arrow[r, mapsto, "\phi(g)"] \arrow[d, mapsto, "\phi(x)"'] & Hg \arrow[d, mapsto, "\phi(x)"] \\
Hx \arrow[r, mapsto, "\phi(g)"] & Hxg = xHg
\end{tikzcd}$$

The G-set automorphism group

Theorem

If G acts on the set $S = H\backslash G$ of right cosets of $H \leq G$, then

$$\mathrm{Aut}_G(S) \cong N_G(H)/H.$$

Proof

We'll apply the FHT to the map

$$f\colon N_G(H) \longrightarrow \mathrm{Aut}_G(S), \qquad x \longmapsto \sigma_x,$$

where $\sigma_x\colon Hg \longmapsto Hxg$.

Homomorphism: Straightforward exercise. ✓

Onto: Immediate from Lemma 2. ✓

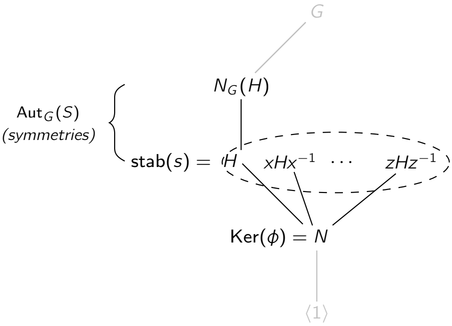
$\mathrm{Ker}(f) = H$. "$\subseteq$":

$$x \in \mathrm{Ker}(\phi) \quad \Leftrightarrow \quad Hg = Hxg,\ \forall\, g \in G \quad \Leftrightarrow \quad H = Hx \quad \Leftrightarrow \quad x \in H.$$

"$\supseteq$": If $h \in H$, then $\sigma_h\colon Hg \mapsto Hhg = Hx$. ✓

The result now follows from the FHT. ☐

G-set automorphisms



G

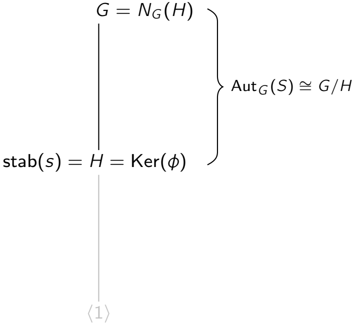
$N_G(H)$

$\text{Aut}_G(S)$ *(symmetries)*

$\text{stab}(s) = H \quad xHx^{-1} \quad \cdots \quad zHz^{-1}$

$\text{Ker}(\phi) = N$

$\langle 1 \rangle$



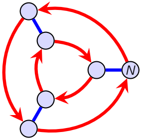
$G = N_G(H)$

$\text{Aut}_G(S) \cong G/H$

$\text{stab}(s) = H = \text{Ker}(\phi)$

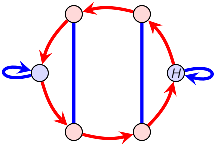
$\langle 1 \rangle$

$N = \langle r^3 \rangle$; *normal*



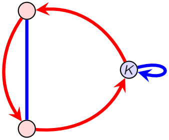
$\text{Aut}_G(N \backslash G) \cong D_3$

$H = \langle f \rangle$; *moderately unnormal*



$\text{Aut}_G(H \backslash G) \cong C_2$

$K = \langle r^3, f \rangle$; *fully unnormal*

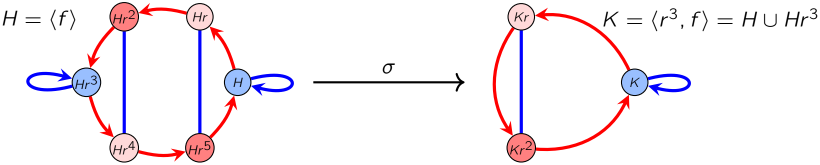


$\text{Aut}_G(K \backslash G) \cong C_1$

G-set homomorphisms

Dropping bijectivity of $\sigma\colon S_1 \to S_2$ defines a G-set homomorphism, or G-equivariant map.

Consider this example of D_6-sets:



This can be described by the following commutative diagram:

$$\begin{tikzcd}
H\backslash G \arrow[r, mapsto, "\phi(g)"] \arrow[d, mapsto, "\sigma"'] & H\backslash G \arrow[d, mapsto, "\sigma"] \\
K\backslash G \arrow[r, mapsto, "\phi(g)"] & K\backslash G
\end{tikzcd}
\qquad
\begin{tikzcd}
Hx \arrow[r, mapsto, "\phi(g)"] \arrow[d, mapsto, "\sigma"'] & Hxg \arrow[d, mapsto, "\sigma"] \\
Kx \arrow[r, mapsto, "\phi(g)"] & Kxg
\end{tikzcd}$$

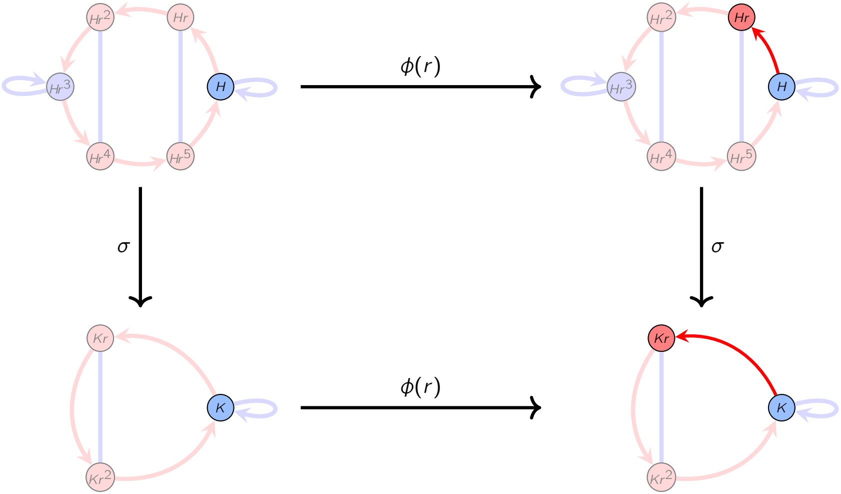
Key idea

We say that "*the map σ commutes with the action of the group.*"

G-set homomorphisms

Here is that example again for $G = D_6$ and subgroups:

$H = \langle f \rangle$ (*moderately unnormal*), $\quad K = \langle r^3, f \rangle = H \cup Hr^3 = N_G(H)$, (*fully unnormal*)

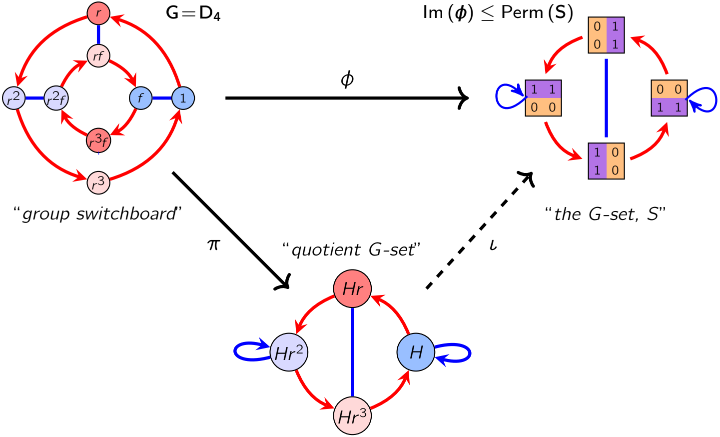


The "fundamental homomorphism theorem for G-sets"

Orbit-stabilizer theorem, restated

If $\phi\colon G \to \mathrm{Perm}(S)$ is a transitive action and $s \in S$, then $\mathrm{orb}(s)$ is isomorphic to the quotient of G by $H = \mathrm{stab}(s)$:

$$H \backslash G \cong \mathrm{orb}(s).$$



A creative application of a group action

Cauchy's theorem

If p is a prime dividing $|G|$, then G has an element (and hence a subgroup) of order p.

Proof

Let P be the set of ordered p-tuples of elements from G whose product is e:

$$(x_1, x_2, \dots, x_p) \in P \quad \text{iff} \quad x_1 x_2 \cdots x_p = e \,.$$

Observe that $|P| = |G|^{p-1}$. (We can choose $x_1, \dots, x_{p-1}$ freely; then x_p is forced.)

The group $\mathbb{Z}_p$ acts on P by cyclic shift:

$$\phi \colon \mathbb{Z}_p \longrightarrow \mathrm{Perm}(P), \qquad (x_1, x_2, \dots, x_p) \xrightarrow{\phi(1)} (x_2, x_3 \dots, x_p, x_1) \,.$$

The set P is partitioned into orbits, each of size $|\,\mathrm{orb}(s)| = [\mathbb{Z}_p : \mathrm{stab}(s)] = 1$ or p.

The only way that the orbit of $(x_1, x_2, \dots, x_p)$ can have size 1 is if $x_1 = \cdots = x_p$.

Clearly, $(e, \dots, e) \in P$ is a fixed point.

The $|G|^{p-1} - 1$ other elements in P sit in orbits of size 1 or p.

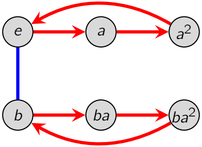
Since $p \nmid |G|^{p-1} - 1$, there must be other orbits of size 1. Thus, some $(x, \dots, x) \in P$, with $x \neq e$ satisfies $x^p = e$. $\square$

Classification of groups of order 6

By Cauchy's theorem, every group of order 6 must have:

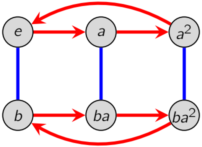
- an element a of order 3
- an element b of order 2.

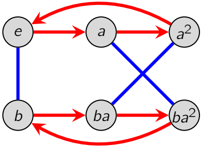
Clearly, $G = \langle a, b \rangle$, and so G must have the following "partial Cayley graph":



It is now easy to see that up to isomorphism, there are only 2 groups of order 6:

$C_6 \cong C_2 \times C_3$





D_3

Exercise. Classify groups of order 8 with a similar argument.

p-groups and the Sylow theorems

Definition

A p-group is a group whose order is a power of a prime p. A p-group that is a subgroup of a group G is a p-subgroup of G.

Notational convention

Throughout, G will be a group of order $|G| = p^n \cdot m$, with $p \nmid m$. That is, p^n is the *highest power* of p dividing $|G|$.

There are three Sylow theorems, and loosely speaking, they describe the following about a group's p-subgroups:

1. **Existence**: In every group, p-subgroups of all possible sizes exist.
2. **Relationship**: All maximal p-subgroups are conjugate.
3. **Number**: Strong restrictions on the number of p-subgroups a group can have.

Together, these place strong restrictions on the structure of a group G with a fixed order.

p-groups

Before we introduce the Sylow theorems, we need to better understand p-groups.

Recall that a p-group is any group of order p^n. Examples, of 2-groups that we've seen include C_1, C_4, V_4, D_4 and Q_8, C_8, $C_4 \times C_2$, D_8, SD_8, Q_{16}, SA_8, DQ_8, . . .

p-group Lemma

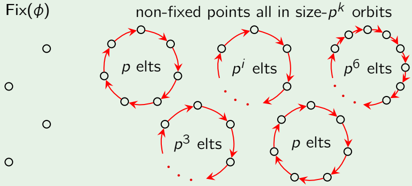
If a p-group G acts on a set S via $\phi\colon G \to \mathrm{Perm}(S)$, then

$$|\operatorname{Fix}(\phi)| \equiv_p |S|.$$

Proof (sketch)

Suppose $|G| = p^n$.

By the orbit-stabilizer theorem, the only possible orbit sizes are $1, p, p^2, \ldots, p^n$.



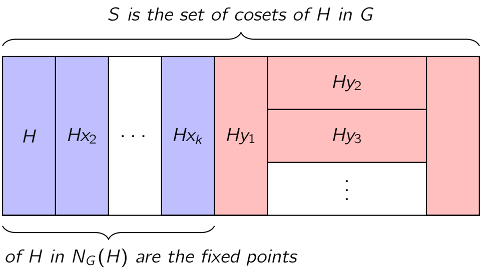
Normalizer lemma, Part 1

If H is a p-subgroup of G, then

$$[N_G(H) \colon H] \equiv_p [G \colon H] \,.$$

Approach:

- Let H (not G!) act on the (right) cosets of H by (right) multiplication.



- Apply our lemma: $|\operatorname{Fix}(\phi)| \equiv_p |S|$.

p-groups

Normalizer lemma, Part 1

If H is a p-subgroup of G, then

$$[N_G(H)\colon H] \equiv_p [G\colon H]\,.$$

Proof

Let $S = H\backslash G = \{Hx \mid x \in G\}$. The group H acts on S by **right-multiplication**, via $\phi\colon H \to \mathrm{Perm}(S)$, where

$$\phi(h) = \text{the permutation sending each } Hx \text{ to } Hxh.$$

The fixed points of ϕ are the cosets Hx in the normalizer $N_G(H)$:

$$\begin{aligned} Hxh = Hx, \quad \forall h \in H \qquad &\Longleftrightarrow \qquad Hxhx^{-1} = H, \quad \forall h \in H \\ &\Longleftrightarrow \qquad xhx^{-1} \in H, \quad \forall h \in H \\ &\Longleftrightarrow \qquad x \in N_G(H)\,. \end{aligned}$$

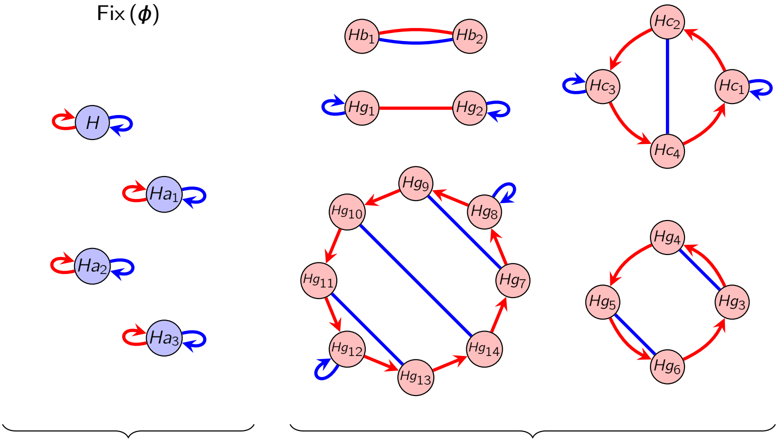
Therefore, $|\,\mathrm{Fix}(\phi)| = [N_G(H)\colon H]$, and $|S| = [G : H]$. By our p-group Lemma,

$$|\,\mathrm{Fix}(\phi)| \equiv_p |S| \quad \Longrightarrow \quad [N_G(H)\colon H] \equiv_p [G\colon H].$$

☐

p-groups

Here is a picture of the action of the p-subgroup H (for $p = 2$) on the set $S = H \backslash G$, from the proof of the normalizer lemma.



The fixed points are the cosets in $N_G(H)$

Cosets not in $N_G(H)$ *are in orbits of order* p^i*, for various* $i \geq 1$

p-subgroups

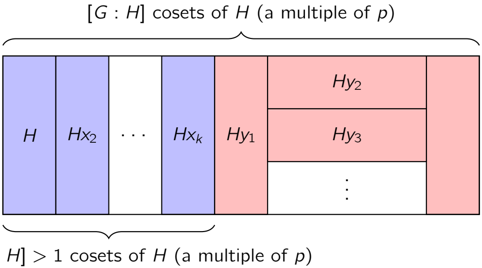
Recall that $H \leq N_G(H)$ (always), and H is fully unnormal if $H = N_G(H)$.

Normalizer lemma, Part 2

Suppose $|G| = p^n m$, and $H \leq G$ with $|H| = p^i < p^n$. Then $H \lneq N_G(H)$, and the index $[N_G(H) : H]$ is a multiple of p.

H is not "*fully unnormal*":

$$H \lneq N_G(H) \leq G$$


$[G : H]$ cosets of H (a multiple of p)

$[N_G(H) : H] > 1$ cosets of H (a multiple of p)

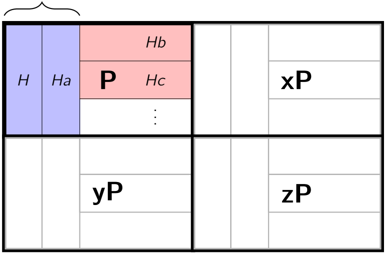
Important corollaries

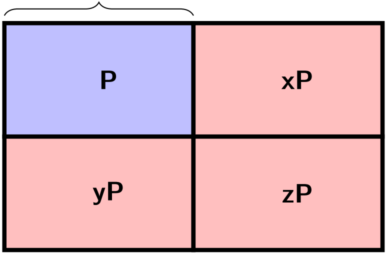
- p-groups cannot have any fully unnormal subgroups (i.e., $H \lneq N_G(H)$).
- In *any* finite group, the only fully unnormal p-subgroups are maximal.

Normalizers of p-subgroups

Let H be properly contained in a maximal p-subgroup $P \lneq G$.

- The normalizer of H *must* grow in P (and hence in G)
- The normalizer of P *need not* grow in G.


$H \lneq N_P(H) \leq N_G(H)$


it may happen that $P = N_G(P)$

Proof of the normalizer lemma

Normalizer lemma, Part 2

Suppose $|G| = p^n m$, and $H \leq G$ with $|H| = p^i < p^n$. Then $H \lneq N_G(H)$, and the index $[N_G(H) : H]$ is a multiple of p.

Proof

Since $H \trianglelefteq N_G(H)$, we can create the quotient map

$$\pi \colon N_G(H) \longrightarrow N_G(H)/H\,, \qquad\qquad \pi \colon g \longmapsto gH\,.$$

The size of the quotient group is $[N_G(H) \colon H]$, the number of cosets of H in $N_G(H)$.

By the normalizer lemma Part 1, $[N_G(H) \colon H] \equiv_p [G \colon H]$. By Lagrange's theorem,

$$[N_G(H) \colon H] \equiv_p [G \colon H] = \frac{|G|}{|H|} = \frac{p^n m}{p^i} = p^{n-i} m \equiv_p 0.$$

Therefore, $[N_G(H) \colon H]$ is a multiple of p, so $N_G(H)$ must be strictly larger than H. ☐

The Sylow theorems

Recall the following question that we asked earlier in this course.

Open-ended question

What group structural properties are possible, what are impossible, and how does this depend on $|G|$?

One approach is to decompose large groups into "building block subgroups." For example:

> *given a group of order* $72 = 2^3 \cdot 3^2$, *what can we say about its* 2*-subgroups and* 3*-subgroups?.*

This is the idea behind the Sylow theorems, developed by Norwegian mathematician Peter Sylow (1832–1918).

The Sylow theorems address the following questions of a finite group G:

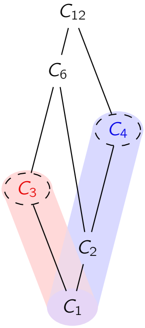
1. How big are its p-subgroups?
2. How are the p-subgroups related?
3. How many p-subgroups are there?
4. Are any of them normal?

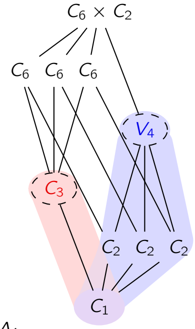
An example: groups of order 12

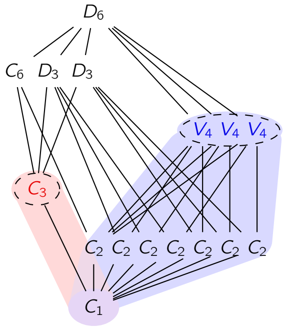
The Sylow theorems can be used to classify all groups of order 12.

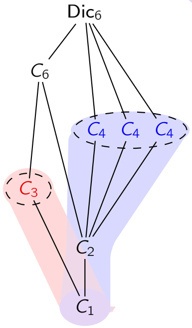
We've already seen them all.

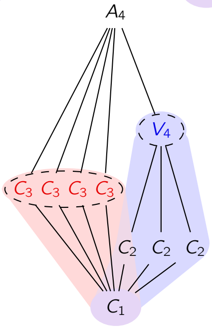
What patterns do you notice about the 2-groups and 3-groups, that might generalize to all p-subgroups?











The Sylow theorems

Notational convention

Througout, G will be a group of order $|G| = p^n \cdot m$, with $p \nmid m$.

That is, p^n is the *highest power* of p dividing $|G|$.

A subgroup of order p^n is called a Sylow p-subgroup.

Let $\mathrm{Syl}_p(G)$ denote the set of Sylow p-subgroups, and $n_p := \big|\mathrm{Syl}_p(G)\big|$.

There are three Sylow theorems, and loosely speaking, they describe the following about a group's p-subgroups:

1. **Existence**: In every group, p-subgroups of all possible sizes exist, and they're "*nested*".
2. **Relationship**: All maximal p-subgroups are conjugate.
3. **Number**: There are strong restrictions on n_p, the number of Sylow p-subgroups.

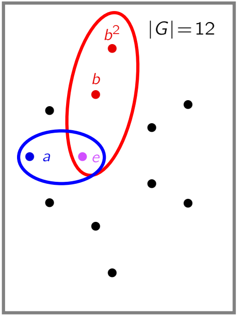
Together, these place strong restrictions on the structure of a group G with a fixed order.

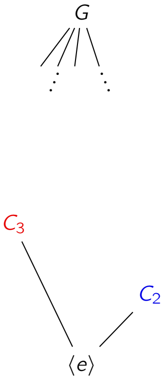
Our unknown group of order 12

Throughout, we will have a running example, a "mystery group" G of order $12 = 2^2 \cdot 3$.

We already know a little bit about G. By Cauchy's theorem, it must have:

- an element a of order 2, and
- an element b of order 3.





Using *only* the fact that $|G| = 12$, we will unconver as much about its structure as we can.

The 1^{st} Sylow theorem: existence of p-subgroups

First Sylow theorem

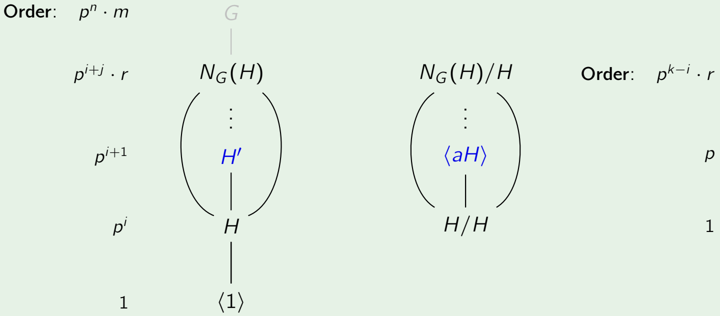
G has a subgroup of order p^k, for each p^k dividing $|G|$.

Also, every non-Sylow p-subgroup sits inside a larger p-subgroup.

Proof

Take any $H \leq G$ with $|H| = p^i < p^n$. We know $H \trianglelefteq N_G(H)$ and p divides $|N_G(H)/H|$.

Find an element aH of order p. The union of cosets in $\langle aH \rangle$ is a subgroup of order p^{i+1}.



Order: $p^n \cdot m$

G

$p^{i+j} \cdot r$ $N_G(H)$ $N_G(H)/H$ **Order**: $p^{k-i} \cdot r$

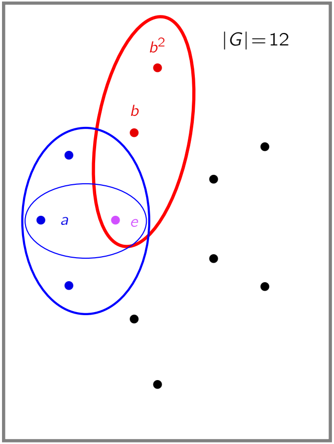
p^{i+1} H' $\langle aH \rangle$ p

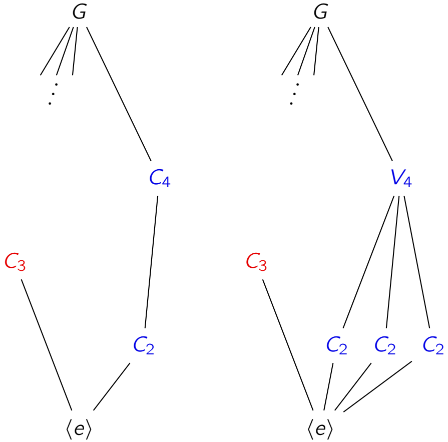
p^i H H/H 1

1 $\langle 1 \rangle$

Our unknown group of order 12

By the first Sylow theorem, $\langle a\rangle$ is contained in a subgroup of order 4, which could be V_4 or C_4, or possibly both.





The 2^{nd} Sylow theorem: relationship among p-subgroups

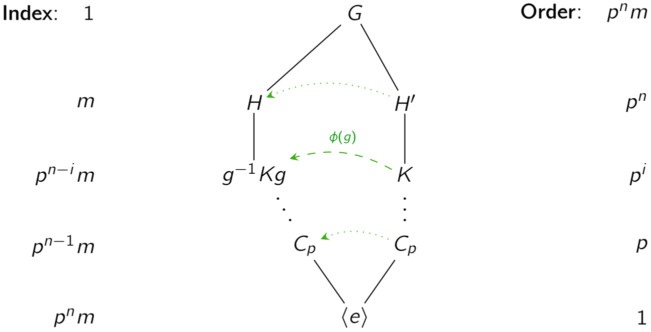
Second Sylow theorem

Any two Sylow p-subgroups are conjugate (and hence isomorphic).

We'll actually prove a stronger version, which easily implies the 2nd Sylow theorem.

Strong second Sylow theorem

Let $H \in \mathrm{Syl}(G)$, and $K \leq G$ any p-subgroup. Then K is conjugate to a subgroup of H.



The 2^{nd} Sylow theorem: All Sylow p-subgroups are conjugate

Strong second Sylow theorem

Let H be a Sylow p-subgroup, and $K \leq G$ any p-subgroup. Then K is conjugate to some subgroup of H.

Proof

Let $S = H\backslash G = \{Hg \mid g \in G\}$, the set of right cosets of H.

The group K acts on S by **right-multiplication**, via $\phi\colon K \to \mathrm{Perm}(S)$, where

$$\phi(k) = \text{the permutation sending each } Hg \text{ to } Hgk.$$

A fixed point of ϕ is a coset $Hg \in S$ such that

$$\begin{aligned} Hgk = Hg\,, \quad \forall k \in K \qquad &\Longleftrightarrow \qquad Hgkg^{-1} = H\,, \quad \forall k \in K \\ &\Longleftrightarrow \qquad gkg^{-1} \in H\,, \quad \forall k \in K \\ &\Longleftrightarrow \qquad gKg^{-1} \subseteq H. \end{aligned}$$

Thus, *if we can show that ϕ has a fixed point Hg, we're done!*

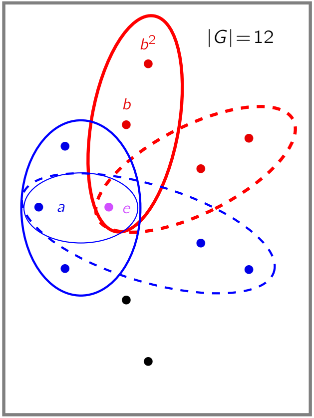
All we need to do is show that $|\mathrm{Fix}(\phi)| \not\equiv_p 0$. By the p-group Lemma,

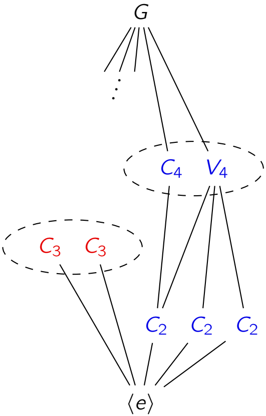
$$|\mathrm{Fix}(\phi)| \equiv_p |S| = [G:H] = m \not\equiv_p 0. \qquad \square$$

Our unknown group of order 12

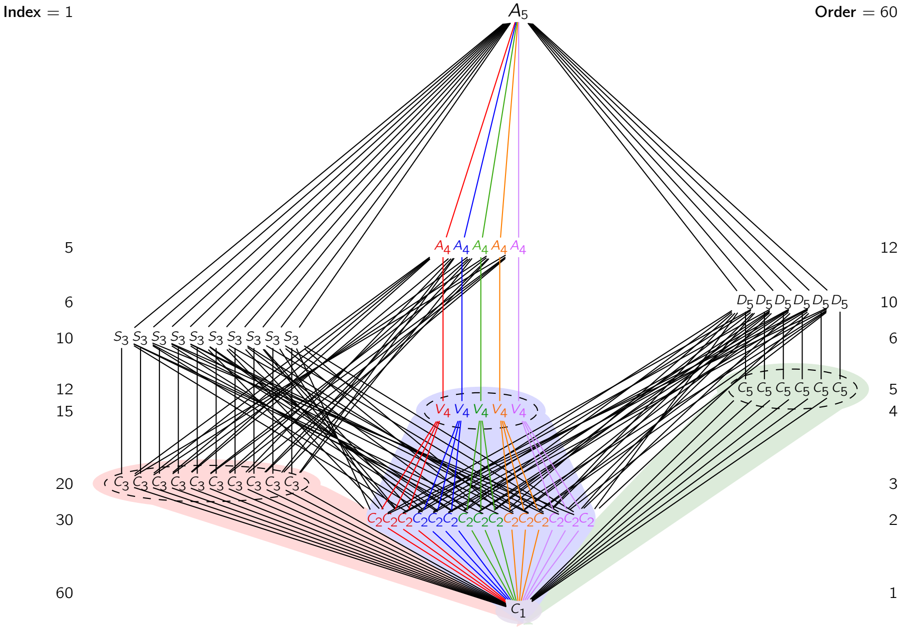
By the second Sylow theorem, all Sylow *p*-subgroups are conjugate, and hence isomorphic.

This eliminates the following subgroup lattice of a group of order 12.





Example: A_5 has no nontrival proper normal subgroups



The normalizer of the normalizer

Notice how in A_5:

- all Sylow p-subgroups are moderately unnormal
- the normalizer of each Sylow p-subgroup is fully unnormal. That is:

$$N_G(N_G(P)) = N_G(P)$$

Proposition

Let P be a non-normal Sylow p-subgroup of G. Then its normalizer is fully unnormal.

Proof

We'll verify the equivalent statement of $N_G(N_G(P)) = N_G(P)$.

Note that P is a normal Sylow p-subgroup of $N_G(P)$.

By the 2nd Sylow theorem, P is the unique Sylow p-subgroup of $N_G(P)$.

Take an element x that normalizes $N_G(P)$ (i.e., $x \in N_G(N_G(P))$. We'll show that it also normalizes P. By definition, $xN_G(P)x^{-1} = N_G(P)$, and so

$$P \leq N_G(P) \qquad \Longrightarrow \qquad xPx^{-1} \leq xN_G(P)x^{-1} = N_G(P).$$

But xPx^{-1} is also a Sylow p-subgroup of $N_G(P)$, and by uniqueness, $xPx^{-1} = P$. ☐

The 3$^{\text{rd}}$ Sylow theorem: number of p-subgroups

Third Sylow theorem

Let n_p be the number of Sylow p-subgroups of G. Then

$$n_p \text{ divides } |G| \qquad \text{and} \qquad n_p \equiv_p 1\,.$$

(Note that together, these imply that $n_p \mid m$, where $|G| = p^n \cdot m$.)

Proof

Take $H \in \mathrm{Syl}_p(G)$. By the 2nd Sylow theorem, $n_p = |\,\mathrm{cl}_G(H)| = [G : N_G(H)] \,\big|\, |G|$. ✓

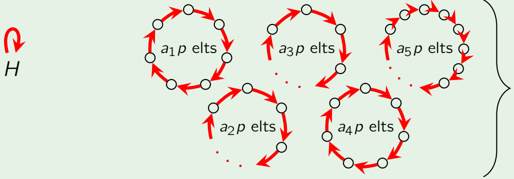
The subgroup H acts on $S = \mathrm{Syl}_p(G)$ by conjugation, via $\phi \colon G \to \mathrm{Perm}(S)$, where

$$\phi(h) = \text{the permutation sending each } K \text{ to } h^{-1}Kh.$$

Goal: *show that H is the unique fixed point.*

$|\,\mathrm{Fix}(\phi)| = 1$

other Sylow p-subgroups are in larger orbits



total # Sylow p-subgroups
$= n_p = |S| \equiv_p |\,\mathrm{Fix}(\phi)|$

The 3^{rd} Sylow theorem: number of p-subgroups

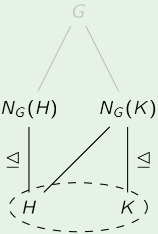
Proof (cont.)

Goal: *show that H is the unique fixed point*.

Let $K \in \text{Fix}(\phi)$. Then $K \leq G$ is a Sylow p-subgroup satisfying

$$h^{-1}Kh = K\,, \quad \forall h \in H \quad \Longleftrightarrow \quad H \leq N_G(K) \leq G\,.$$

- H and K are p-Sylow in G, and in $N_G(K)$.
- H and K are conjugate in $N_G(K)$. (2nd Sylow thm.)
- $K \trianglelefteq N_G(K)$, thus is only conjugate to itself in $N_G(K)$.

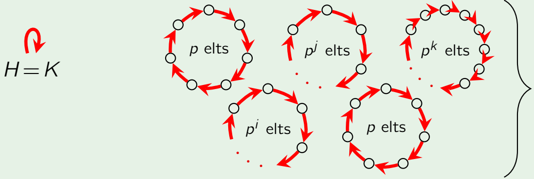


Thus, $K = H$. That is, $\text{Fix}(\phi) = \{H\}$.

By the p-group Lemma, $n_p := |S| \equiv_p |\text{Fix}(\phi)| = 1$. ☐

$|\text{Fix}(\phi)| = 1$

other Sylow p-subgroups are in larger orbits



total # Sylow p-subgroups
$= n_p = |S| \equiv_p |\text{Fix}(\phi)| = 1$

Summary of the proofs of the Sylow theorems

For the 1st Sylow theorem, we started with $H = \{e\}$, and inductively created larger subgroups of size $p, p^2, \dots, p^n$.

For the 2^{nd} and 3^{rd} Sylow theorems, we used a clever group action and then applied one or both of the following:

(i) *orbit-stabilizer theorem*. If G acts on S, then $|\,\mathbf{orb}(s)|\cdot|\,\mathbf{stab}(s)| = |G|$.

(ii) *p-group lemma*. If a p-group acts on S, then $|S| \equiv_p |\,\mathrm{Fix}(\phi)|$.

To summarize, we used:

S2 The action of $K \in \mathrm{Syl}_p(G)$ on $S = H\backslash G$ by right multiplication for some other $H \in \mathrm{Syl}_p(G)$.

S3a The action of G on $S = \mathrm{Syl}_p(G)$, by conjugation.

S3b The action of $H \in \mathrm{Syl}_p(G)$ on $S = \mathrm{Syl}_p(G)$, by conjugation.

Our mystery group order 12

By the 3rd Sylow theorem, every group G of order $12 = 2^2 \cdot 3$ must have:

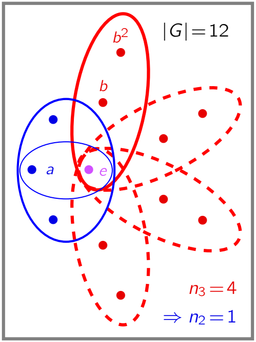
- n_3 Sylow 3-subgroups, each of order 3.

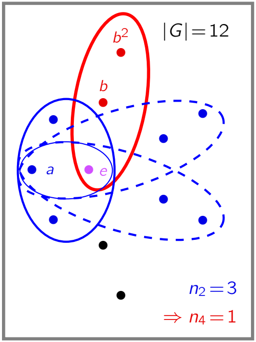
$$n_3 \mid 4, \qquad n_3 \equiv 1 \pmod 3 \qquad \Longrightarrow \qquad n_3 = 1 \text{ or } 4.$$

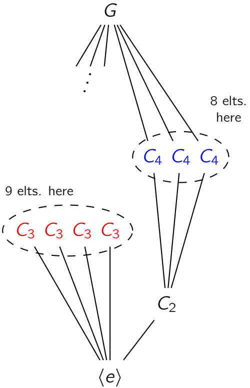
- n_2 Sylow 2-subgroups of order $2^2 = 4$.

$$n_2 \mid 3, \qquad n_2 \equiv 1 \pmod 2 \qquad \Longrightarrow \qquad n_2 = 1 \text{ or } 3.$$

But both are not possible! (There aren't enough elements.)



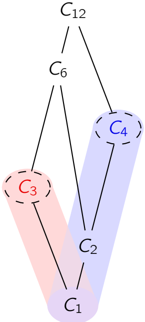


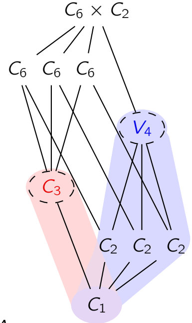


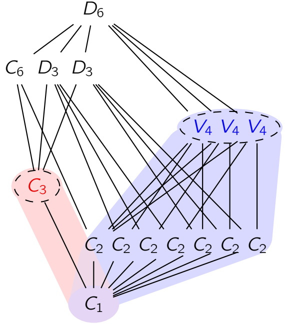
Classification of groups of order 12

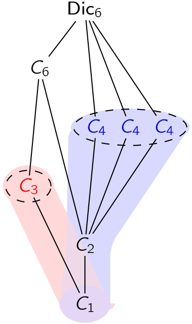
$P = C_4$ or V_4, $Q = C_3$

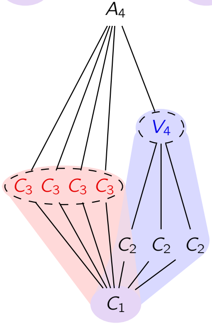
- **Case 1**: $n_2 = 1$, $n_3 = 1$.
- **Case 2**: $n_2 = 1$, $n_3 = 4$.
- **Case 3**: $n_2 = 3$, $n_3 = 1$.











Simple groups and the Sylow theorems

Definition

A group G is simple if its only normal subgroups are G and $\langle e \rangle$.

Simple groups are to groups what primes are to integers, and are essential to understand.

The Sylow theorems are very useful for establishing statements like:

"There are no simple groups of order k (for some k)."

Since all Sylow p-subgroups are conjugate, the following result is immediate.

Remark

A Sylow p-subgroup is normal in G iff it's the unique Sylow p-subgroup (that is, if $n_p = 1$).

Thus, if we can show that $n_p = 1$ for some p dividing $|G|$, then G cannot be simple.

For some $|G|$, this is harder than for others, and sometimes it's not possible.

Tip

When trying to show that $n_p = 1$, it's usually helpful to analyze the largest primes first.

An easy example

We'll see three examples of showing that groups of a certain size cannot be simple, in successive order of difficulty.

Then we'll see several that I will leave as homework.

Proposition

There are no simple groups of order 84.

Proof

Since $|G| = 84 = 2^2 \cdot 3 \cdot 7$, the third Sylow theorem tells us:

- n_7 divides $2^2 \cdot 3 = 12$ (so $n_7 \in \{1, 2, 3, 4, 6, 12\}$)
- $n_7 \equiv_7 1$.

The only possibility is that $n_7 = 1$, so the Sylow 7-subgroup must be normal. $\square$

Observe why it is beneficial to use the largest prime first:

- n_3 divides $2^2 \cdot 7 = 28$ and $n_3 \equiv_3 1$. Thus $n_3 \in \{1, \cancel{2}, 4, 7, \cancel{14}, 28\}$.
- n_2 divides $3 \cdot 7 = 21$ and $n_2 \equiv_2 1$. Thus $n_2 \in \{1, 3, 7, 21\}$.

A harder example

Proposition

There are no simple groups of order 351.

Proof

Since $|G| = 351 = 3^3 \cdot 13$, the third Sylow theorem tells us:

- n_{13} divides $3^3 = 27$ (so $n_{13} \in \{1, 3, 9, 27\}$)
- $n_{13} \equiv_{13} 1$.

The only possibilies are $n_{13} = 1$ or 27.

A Sylow 13-subgroup P has order 13, and a Sylow 3-subgroup Q has order $3^3 = 27$. Therefore, $P \cap Q = \{e\}$.

Suppose $n_{13} = 27$. Every Sylow 13-subgroup contains 12 non-identity elements, and so G must contain $27 \cdot 12 = 324$ elements of order 13.

This leaves $351 - 324 = 27$ elements in G not of order 13. Thus, G contains only one Sylow 3-subgroup (i.e., $n_3 = 1$) and so G cannot be simple. ☐

The hardest example

Proposition

There are no simple groups of order $24 = 2^3 \cdot 3$.

From the 3rd Sylow theorem, we can only conclude that $n_2 \in \{1, 3\}$ and $n_3 = \{1, 4\}$.

Let H be a Sylow 2-subgroup, which has relatively "small" index: $[G : H] = 3$.

Lemma

If G has a subgroup of index $[G : H] = n$, and $|G|$ does not divide $n!$, then G is not simple.

Proof

Let G act on the **right cosets** of H (i.e., $S = H \backslash G$) by **right-multiplication**:

$$\phi \colon G \longrightarrow \mathrm{Perm}(S) \cong S_n\,, \qquad \phi(g) = \text{the permutation that sends each } Hx \text{ to } Hxg.$$

Recall that $\mathrm{Ker}(\phi) \trianglelefteq G$, and is the intersection of all conjugate subgroups of H:

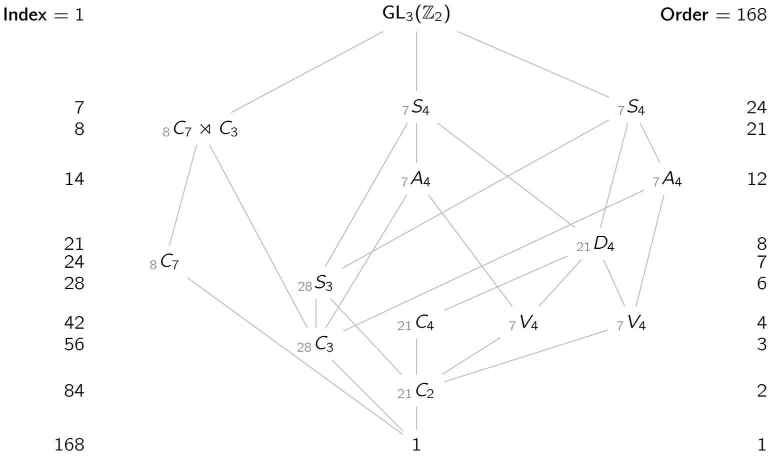
$$\langle e \rangle \leq \mathrm{Ker}(\phi) = \bigcap_{x \in G} x^{-1} H x \lneq G$$

If $\mathrm{Ker}(\phi) = \langle e \rangle$ then $\phi \colon G \hookrightarrow S_n$ is an embedding, which is impossible because $|G| \nmid n!$. ☐

The second smallest non-abelian simple group

Exercise (HW)

Show that the simple group $G = \mathrm{GL}_3(\mathbb{Z}_2)$ of order 168 is a subgroup of A_8.



Index $= 1$ $\mathrm{GL}_3(\mathbb{Z}_2)$ **Order** $= 168$

7 ${}_7S_4$ ${}_7S_4$ 24

8 ${}_8C_7 \rtimes C_3$ 21

14 ${}_7A_4$ ${}_7A_4$ 12

21 ${}_{21}D_4$ 8

24 ${}_8C_7$ 7

28 ${}_{28}S_3$ 6

42 ${}_{21}C_4$ ${}_7V_4$ ${}_7V_4$ 4

56 ${}_{28}C_3$ 3

84 ${}_{21}C_2$ 2

168 1 1

A_6: the third smallest non-abelian simple group

Exercise (HW)

Prove that there are no simple groups of order $90 = 2 \cdot 3^2 \cdot 5$.