Difference Equations

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Motivation: Population dynamics

Consider a population of insects that reproduces daily, of size $P(t)$:

- *birth rate* is $\beta \in [0, \infty)$,
- *death rate* is $\delta \in [0, 1]$.

This can be modeled by a simple equation:

$$\Delta P = \beta P - \delta P = (\beta - \delta)P\,.$$

Suppose time is *discretized*, e.g., it only takes integer values: $t = 0, 1, 2, \ldots$.

Let $P_t = P(t) =$ population at time t.

Then $\Delta P = P_{t+1} - P_t$, from which it follows that

$$P_{t+1} = P_t + \Delta P = P_t + (\beta - \delta)P_t = (1 + \beta - \delta)P_t\,.$$

Letting $\lambda = 1 + \beta - \delta$ (the "*finite growth rate*"), we can write this as $\boxed{P_{t+1} = \lambda P_t}$.

An example

Consider a population of insects that reproduces daily, with the following parameters:

- *initial population* $P_0 = 300$,
- *birth rate* $\beta = .03$,
- *death rate* $\delta = .01$.

Then the finite growth rate is $\lambda = 1 + \beta - \delta = 1.02$, and

$$\begin{aligned} P_1 &= (1.02)P_0 \\ P_2 &= (1.02)P_1 = (1.02)^2 P_0 \\ P_3 &= (1.02)P_2 = (1.02)^3 P_0 \\ &\vdots \end{aligned}$$

The closed-form solution $P_t = \lambda^t P_0$. This is called exponential growth.

What is a difference equation?

Definition

Let Q be a quantity defined for all $t \in \mathbb{N}$, such that $Q_{t+1} = F(Q_t)$ for some function.

In the previous example: $F(x) = \lambda x$. This is the Malthusian model, from 1798.

It is a linear difference equation because $F(x)$ is linear.

Let's compare difference equations to differential equations:

- Difference equations are *discrete time, continuous space*.
- Differential equations are *continuous time, continuous space*.

The differential equation form of the Malthusian model is $P' = \lambda P$.

Exercise

Can you think of a model that is discrete time and discrete space? Or continuous time and discrete space?

Which type of model to use?

Broad goals

- Find an appropriate model.
- Analyze models that naturally arise.

For example, consider the following three problems to be modeled:

1. Let P be a population of $P_0 = 300$ insects with birth rate $\beta = .03$ and death rate $\delta = .01$.
2. Let P be the value of an initial investment of $P_0 = 300$ dollars with fixed 2% interest rate, i.e., $\lambda = 1.02$.
3. Let P be a mass of a population of bacteria that is initially $P_0 = 300$ grams, with growth rate insects with finite growth rate $\lambda = 1.02$.

Exercise

Which of these are more suited for difference equations, and which for differential equations?

Logistic equation for population growth

Realistically, a population's growth rate isn't constant – it depends on size. ("*density dependent*").

Big idea

Analyze $\Delta P/P =$ per capita growth rate.

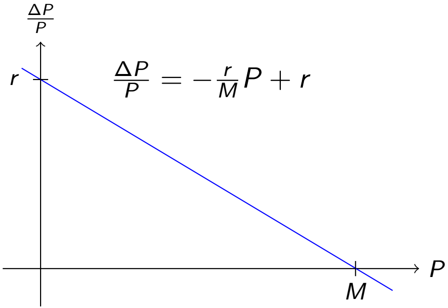
- P small: $\frac{\Delta P}{P}$ large.
- P large: $\frac{\Delta P}{P}$ small.
- P too large: $\frac{\Delta P}{P} < 0$.

Assumptions:

- Let r be the growth rate when $P = 0$. [Technically, $r = \lim_{P \to 0^+} \frac{\Delta P}{P}$.] This is called the finite intrinsic growth rate.
- Let M be the population for which $\frac{\Delta P}{P} = 0$. This is called the carrying capacity.
- Suppose the growth rate decreases *linearly* with P.

This leads to the logistic model of Pierre Francois Verhulst in 1838.

Logistic equation for population growth



Since the growth rate decreases *linearly* with P, basic algebra gives

$$\frac{\Delta P}{P} = -\frac{r}{M}P + r = r\left(1 - \frac{P}{M}\right).$$

Logistic equation for population growth

Substituting $\Delta P = P_{t+1} - P_t$ into $\frac{\Delta P}{P} = r\left(1 - \frac{P}{M}\right)$, followed by easy algebra yields the discrete logistic model:

$$P_{t+1} = P_t\left(1 + r\left(1 - \frac{P_t}{M}\right)\right).$$

Model validation

To see if this model is reasonable, the first thing to check are some simple cases:

- $P \ll M \Longrightarrow 1 - \frac{P}{M} \approx 1 \Longrightarrow P_{t+1} \approx (1+r)P_t$. [Exponential growth!]
- $P \approx M \Longrightarrow 1 - \frac{P}{M} \approx 0 \Longrightarrow P_{t+1} \approx P_t$.

Exercise

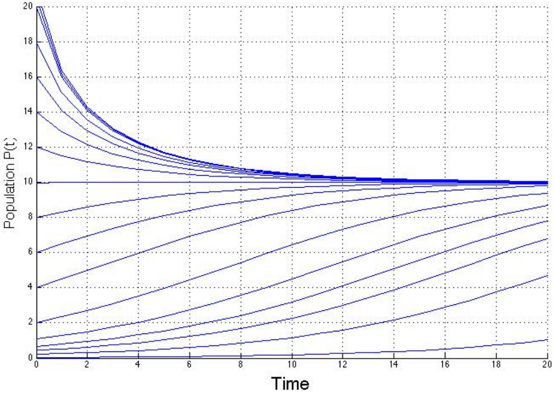
What is $F(x)$ in the discrete logistic model? [It must satisfy $P_{t+1} = F(P_t)$.]

Solutions of difference equations

Difference equations, though simiple, often have *no closed form solution* for P_t.

However, we can plot the solutions for various initial values P_0.

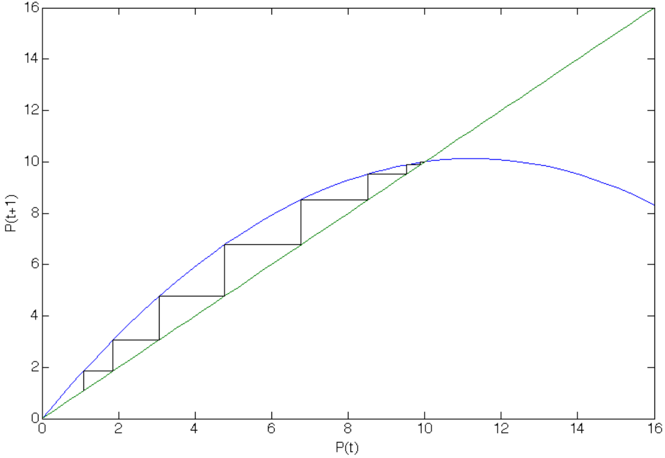
Here are some solutions to the equation $P_{t+1} = P_t + .2P_t(1 - \frac{P_t}{10})$.



Cobwebbing

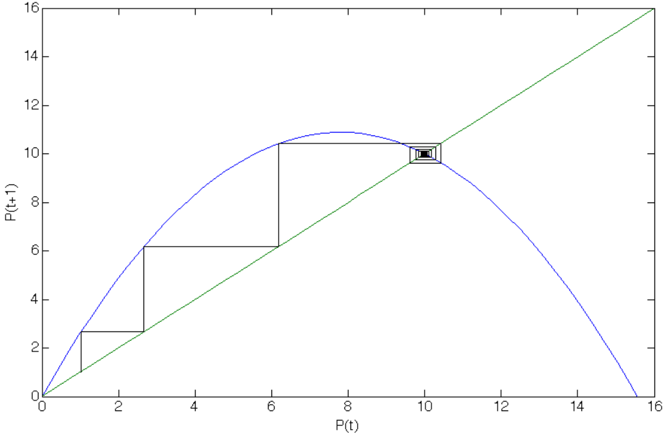
Consider the difference equation $\Delta P = 0.8P_t\left(1 - \frac{P_t}{10}\right)$. Or equivalently, $P_{t+1} = F(P_t) = P_t + 0.8P_t\left(1 - \frac{P_t}{10}\right)$.

We can numerically find $P_0, P_1, P_2, \ldots$ by plotting $F(x) = x + 0.8x(1 - \frac{x}{10})$ and $y = x$ on the same axes, and then by "cobwebbing":



Cobwebbing

Consider another difference equation: $\Delta P = 1.8 P_t \left(1 - \frac{P_t}{10}\right)$. Or equivalently,
$P_{t+1} = F(P_t) = P_t + 1.8 P_t \left(1 - \frac{P_t}{10}\right)$.



Cobwebbing

Questions

1. Sketch a plot of several solution curves $P(t)$ for the difference equations in the previous two examples.
2. What does the spiraling behavior of this cobweb imply about the population $P(t)$?
3. How does this relate to mass-spring systems? [*Hint*: Think about damping.]
4. What features about a population are highlighted in the logistic equation using difference equations that do not arise using differential equations?