Predator-prey models

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Introduction

Consider a population of two species, e.g., foxes ("predator") and rabbits ("prey").

- P_t = size of prey.
- Q_t = size of predator.

The change in population size of each is a function of *both* population sizes:

$$\Delta P = F(P,Q)\,, \qquad \Delta Q = G(P,Q)\,.$$

Question

What would happen if the predator or the prey disappeared?

- Prey, *without* predators: $\Delta P = r(P(1 - P/M))$.
- Predators, *without* prey: $\Delta Q = -uQ$, where $u \in (0,1)$ is per-capita death rate.

Simple predator-prey model

$$\begin{cases} \Delta P = rP(1 - P/M) - sPQ \\ \Delta Q = -uQ + vPQ \end{cases} \qquad r, s, u, v, K > 0,\ u < 1$$

Predator-prey model

Alternate form

$$\begin{cases} P_{t+1} = P_t(1 + r(1 - P_t/M)) - sP_tQ_t \\ Q_{t+1} = (1-u)Q_t + vP_tQ_t \end{cases} \qquad r, s, u, v, K > 0, \quad u < 1$$

The $-sPQ$ and vPQ are called *mass-action terms*. Roughly speaking:

- $-sPQ$ describes a *negative* effect of the predator-prey interaction on the prey,
- vPQ describes a *positive* effect of the predator-prey interaction on the predator.

Qualitatively, larger values of s and v indicate stronger predator-prey interaction.

We can plot the solutions of these equations several ways:

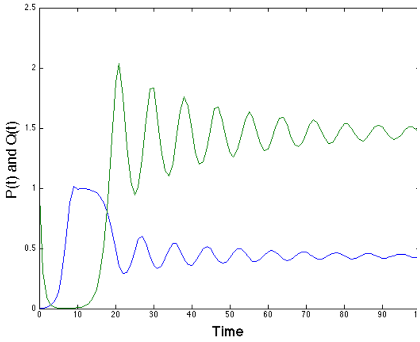
- time plots: P_t vs. t, and Q_t vs. t
- phase plots: Q_t vs. P_t.

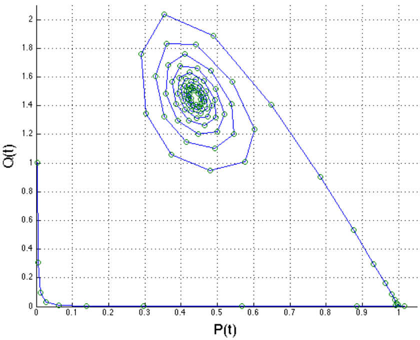
Time plots and phase plots

Consider the following predator-prey model:

$$\begin{cases} P_{t+1} = P_t(1 + 1.3(1 - P_t)) - .5P_tQ_t \\ Q_{t+1} = .3Q_t + 1.6P_tQ_t \end{cases}$$

Solutions can be graphed using a *time plot* (left) or a *phase plot* (right):





Equilibria

To find steady-state population(s), we set $P_t = P_{t+1} = P^*$ and $Q_t = Q_{t+1} = Q^*$.

$$\begin{cases} P_{t+1} = P_t(1 + 1.3(1 - P_t)) - .5P_tQ_t \\ Q_{t+1} = .3Q_t + 1.6P_tQ_t \end{cases} \rightsquigarrow \begin{cases} P^* = P^*(1 + 1.3(1 - P^*)) - .5P^*Q^* \\ Q^* = .3Q^* + 1.6P^*Q^* \end{cases}$$

Via simple algebra, this reduces to the following system

$$\begin{cases} 0 = P^*(1.3 - 1.3P^* - .5Q^*) \\ 0 = Q^*(-.7 + 1.6P^*) \end{cases}$$

If $Q^* = 0$, then $P^* = 0$ or $P^* = 1$.

Alternatively, $P^* = .4375$, which would force $Q^* = 1.4625$.

Thus, there are three equilibria:

$$(P^*, Q^*) = (0, 0), \;\; (1, 0), \;\; (.4375, 1.4625).$$

Equilibria and nullclines

For the general predator-prey model:

$$\begin{cases} P_{t+1} = P_t(1 + r(1 - P_t/M)) - sP_tQ_t \\ Q_{t+1} = (1-u)Q_t + vP_tQ_t \end{cases} \qquad r, s, u, v, M > 0, \quad u < 1$$

the equilibrium equations (set $P_t = P_{t+1} = P^*$ and $Q_t = Q_{t+1} = Q^*$) are

$$\begin{cases} 0 = P^*(r(1-P^*) - sQ^*) \\ 0 = Q^*(-u + vP^*). \end{cases}$$

For Equation 2 to be satisfied, $Q^* = 0$ or $-u + vP^* = 0$.

Furthermore, Equation 1 is satisfied if $P^* = 0$ or $r(1-P^*) - sQ^* = 0$.

By simple algebra, we get three equilibria:

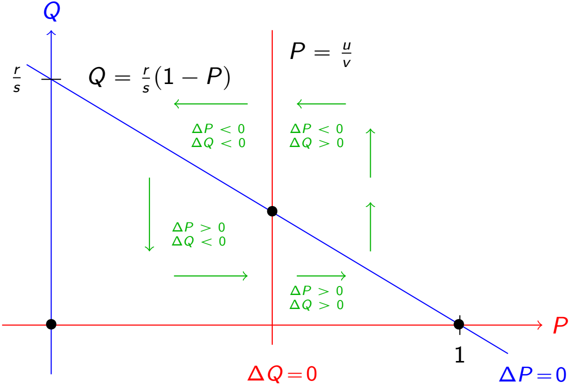
$$(P^*, Q^*) = (0,0), \ \ (1,0), \ \ \left(\frac{u}{v}, \frac{r}{s}\Big(1 - \frac{u}{v}\Big)\right).$$

A *nullcline* is a line on which either $\Delta P = 0$ or $\Delta Q = 0$. In our example:

$$P = 0, \qquad Q = \frac{r}{s}(1-P), \qquad Q = 0, \qquad P = \frac{u}{v}.$$

Nullclines

We can plot the nullclines on the PQ-plane to help visualize the dynamics.



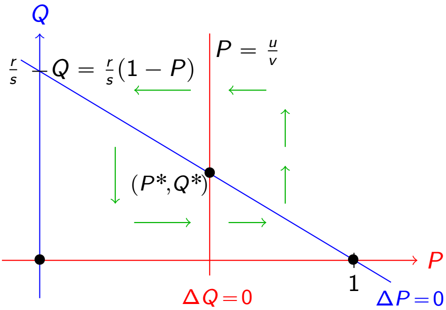
- $\Delta P > 0$ occurs *below* $Q = \frac{r}{s}(1-P)$.
- $\Delta Q > 0$ occurs *to the right of* $P = \frac{u}{v}$.

Do you see how we determine the direction of the green arrows? Can we tell whether it spirals inward or outward?

Nullclines

Remark

Changing r or s doesn't affect the Q-nullcline.



Suppose the predator was an insect and the prey was an agricultural crop.

One might want to introduce a new crop variety with higher r, to try to "outgrow" the predator.

Unfortunately, this won't work: P^* is unchanged, but Q^* increases. (Why?)

Linearization

Suppose (P^*, Q^*) is a fixed point whose stability we wish to understand.

We can plug the following "perturbation" back into the original system:

$$P_t = P^* + p_t\,, \qquad P_{t+1} = P^* + p_{t+1}\,, \qquad Q_t = Q^* + q_t\,, \qquad Q_{t+1} = Q^* + q_{t+1}\,.$$

Consider the fixed point $(P^*, Q^*) = (.4375, 1.4625)$ of our previous example. Plugging

$$P_t = .4375 + p_t\,, \quad P_{t+1} = .4375 + p_{t+1}\,, \quad Q_t = 1.4625 + q_t\,, \quad Q_{t+1} = 1.4625 + q_{t+1}\,.$$

into
$$\begin{cases} P_{t+1} = P_t(1 + 1.3(1 - P_t)) - .5P_tQ_t \\ Q_{t+1} = .3Q_t + 1.6P_tQ_t \end{cases}$$

and simplifying yields

$$\begin{cases} p_{t+1} = .43125p_t - .21875q_t - 1.3p_t^2 - .5p_tq_t \\ q_{t+1} = 2.34p_t + q_t + 1.6p_tq_t \end{cases}$$

For small perturbations (p_t, q_t), we can neglect the nonlinear terms (e.g., p_t^2, q_t^2, and p_tq_t) which are ≈ 0, leaving a linear system $\mathrm{p}_{t+1} \approx A\mathrm{p}_t$.

Linearization (cont.)

Thus, given a small perturbation (p_t, q_t) at time t, it can be described at time $t+1$ by a linear equation $\mathrm{p}_{t+1} \approx \mathrm{A}\mathrm{p}_t$:

$$\begin{bmatrix} p_{t+1} \\ q_{t+1} \end{bmatrix} \approx \begin{bmatrix} .43125 & -.21875 \\ 2.34 & 1 \end{bmatrix} \begin{bmatrix} p_t \\ q_t \end{bmatrix}.$$

The eigenvalues of A are $\lambda = .7156 \pm .6565i$, which have norm

$$|\lambda| = \sqrt{(.7156)^2 + (.6565)^2} = .9711 < 1\,.$$

Thus, this perturbation from the steady-state is shrinking. The population will spiral back into the steady-state $(P^*, Q^*) = (.4375, 1.4625)$.

Types of equilibrium points

- $|\lambda_1| < 1$, $|\lambda_2| < 1$, stable
- $|\lambda_1| > 1$, $|\lambda_2| > 1$, unstable
- $|\lambda_1| < 1$, $|\lambda_2| > 1$, saddle

Other interaction models

- Competition: 2 species fill the same niche in an environment.

$$\begin{cases} \Delta P = rP(1-(P+Q)/M) \\ \Delta Q = rQ(1-(P+Q)/M) \end{cases}$$

Question: Does one species "win"? Or can they co-exist?

- Competition with predator/prey: $\begin{cases} \Delta P = rP(1-(P+Q)/M) - sPQ \\ \Delta Q = rQ(1-(P+Q)/M) \pm vPQ \end{cases}$

- Mutualism: e.g., P = sharks, Q = feeder fish. $\begin{cases} \Delta P = rP(1-P/M) + sPQ \\ \Delta Q = -uQ + vPQ \end{cases}$

- Immune system vs. infective agent:

P : immune cells
Q : level of infection

$$\begin{cases} \Delta P = rQ - sPQ \\ \Delta Q = uQ - vPQ \end{cases}$$

  - $-sPQ$: negative effect on immune system from fighting
  - $-sPQ$: limited effect on immune system from fighting
  - rQ: immune response is proportional to infection level