Linear models of structured populations

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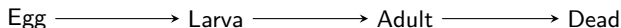http://www.math.clemson.edu/~macaule/

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Motivation: Population dynamics

Consider a population divided into several groups, such as

- children and adults
- egg, larva, pupa, adult

For example, consider a population of insects

$$\text{Egg} \longrightarrow \text{Larva} \longrightarrow \text{Adult} \longrightarrow \text{Dead}$$
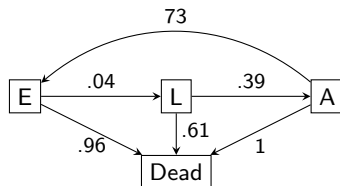
$E_t = \#$ eggs at time t

$L_t = \#$ larve at time t

$A_t = \#$ adults at time t

An example

Suppose we have the following data:

- 4% of eggs survive to become larvae
- 39% of larvae make it to adulthood
- The average adult produces 73 eggs each
- Each adult dies after 1 day

We can write this as a system of difference equations:

$$\begin{cases} E_{t+1} = 73A_t \\ L_{t+1} = .04E_t \\ A_{t+1} = .39L_t \end{cases} \qquad \begin{bmatrix} 0 & 0 & 73 \\ .04 & 0 & 0 \\ 0 & .39 & 0 \end{bmatrix} \begin{bmatrix} E_t \\ L_t \\ A_t \end{bmatrix} = \begin{bmatrix} E_{t+1} \\ L_{t+1} \\ A_{t+1} \end{bmatrix}.$$

By back-substitution, or inspection, we can deduce the following:

$$A_{t+3} = (.39)(.04)(73)A_t = 1.1388A_t$$

Thus, this is just exponential growth. But what if instead of dying, 65% of adults survive another day?

A slighly more complicated example

Suppose we have the following data:

- 4% of eggs survive to become larvae
- 39% of larvae make it to adulthood
- The average adult produces 73 eggs each
- Each day, 35% of adults die.
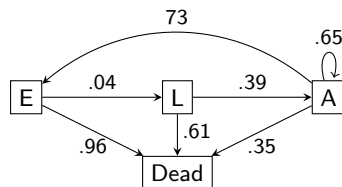
This yields a more complicated system of difference equations:

$$\begin{cases} E_{t+1} = 73A_t \\ L_{t+1} = .04E_t \\ A_{t+1} = .39L_t + .65A_t \end{cases} \qquad \begin{bmatrix} 0 & 0 & 73 \\ .04 & 0 & 0 \\ .65 & .39 & 0 \end{bmatrix} \begin{bmatrix} E_t \\ L_t \\ A_t \end{bmatrix} = \begin{bmatrix} E_{t+1} \\ L_{t+1} \\ A_{t+1} \end{bmatrix}.$$

Questions

- Best way to solve this?
- What is the growth rate?
- What is the long-term behavior?
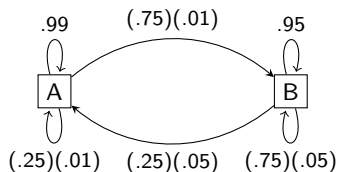- How much effect does changing the initial conditions have?

Another example

Consider a forest that has 2 species of trees, A and B. Let A_t and B_t denote the population of each, in year t.

When a tree dies, a new tree grows in its place (either species).

Each year:

- 1% of the A-trees die
- 5% of the B-trees die
- 25% of the vacant spots go to species A
- 75% of the vacant spots go to species B

This can be written as a 2×2 system:

$$\begin{cases} A_{t+1} = .99A_t + (.25)(.01)A_t + (.25)(.05)B_t \\ B_{t+1} = .95B_t + (.75)(.01)A_t + (.75)(.05)B_t \end{cases}$$

$$\begin{bmatrix} .9925 & .0125 \\ .0075 & .9875 \end{bmatrix} \begin{bmatrix} A_t \\ B_t \end{bmatrix} = \begin{bmatrix} A_{t+1} \\ B_{t+1} \end{bmatrix}$$

Solving systems of difference equations

One way to solve $x_{t+1} = Px_t$:

$$x_1 = Px_0$$
$$x_2 = Px_1 = P(Px_0) = P^2 x_0$$
$$x_3 = Px_2 = P^3 x_0$$
$$\vdots$$

A better method

Find the *eigenvalues* and *eigenvectors* of P.

Then write the initial vector x_0 using a *basis of eigenvectors*.

Suppose $x_0 = c_1 v_1 + c_2 v_2$. Then

$$x_1 = Px_0 = P(c_1 v_1 + c_2 v_2) = c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2$$
$$x_2 = Px_1 = P^2 x_0 = P(c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2) = c_1 \lambda_1^2 v_1 + c_2 \lambda_2^2 v_2\,.$$
$$\vdots$$
$$x_t = P^t x_0 = c_1 \lambda_1^t v_1 + c_2 \lambda_2^t v_2\,.$$

An example, revisted

Let us revisit our "tree example", where $P = \begin{bmatrix} .9925 & .0125 \\ .0075 & .9875 \end{bmatrix}$.

The eigenvalues and eigenvectors of P are

$$\lambda_1 = 1, \ \ v_1 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}, \qquad \lambda_2 = .98, \ \ v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Consider the initial condition $x_0 = \begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = \begin{bmatrix} 10 \\ 990 \end{bmatrix}$.

First step

Write $x_0 = c_1 \begin{bmatrix} 5 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, i.e., solve $Pc = x_0$:

$$\begin{bmatrix} 5 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 990 \end{bmatrix}.$$

$$c = P^{-1}x_0 = -\frac{1}{8} \begin{bmatrix} -1 & -1 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} 10 \\ 990 \end{bmatrix} = \begin{bmatrix} 125 \\ -615 \end{bmatrix}$$

Thus, our initial vector is $x_0 = \begin{bmatrix} 10 \\ 990 \end{bmatrix} = 125 \begin{bmatrix} 5 \\ 3 \end{bmatrix} - 615 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

An example (cont.)

In our example, $x_0 = 125 \begin{bmatrix} 5 \\ 3 \end{bmatrix} - 615 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and so

$$x_t = 125(1)^t \begin{bmatrix} 5 \\ 3 \end{bmatrix} - 615(.98)^t \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 625 - (615)(.98)^t \\ 375 + (615)(.98)^t \end{bmatrix} \,.$$

The long-term behavior of this system is

$$\lim_{t \to \infty} x_t = 125 \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 625 \\ 375 \end{bmatrix} \,.$$

Notice that this does *not* depend on x_0!