Problem 1. Rewrite each of the following sentences so it has either the form *"If P, then Q,"* or *"P if and only if Q,"* without changing the meaning.

1. A group is abelian whenever all nonidentity elements have order 2.

2. Whenever a subset of $\mathbb{R}^n$ is compact, it is closed and bounded.

3. In order for X to be complete, it is necessary that every Cauchy sequence converges.

4. A ring that is a finite integral domain is necessarily a field.

5. Being orthogonal is a sufficient condition for a matrix to be invertible.

6. Being prime is enough to ensure that a nonzero element in $\mathbb{Z}[\sqrt{-5}]$ is irreducible.

7. A sequence converges only if it is Cauchy.

8. For a series $\sum a_n$ to diverge, it suffices that $a_n \not\to 0$.

9. A set is open provided its complement is closed.

10. Every Cauchy sequence in X converges unless X is not complete.

11. A matrix is invertible precisely when its determinant is nonzero.

12. Every finite subgroup of a field's multiplicative group is cyclic.

13. Without compactness of X, continuity of f need not imply boundedness of f.

14. A matrix does not have an orthogonal set of eigenvectors unless it is symmetric.

Problem 2. Translate the following English sentences into symbolic sentences with quantifiers. The universe U for each is given in parentheses.

(a) For every positive number x, there is a unique number y such that $2^y = x$. $(U = \mathbb{R})$

(b) For every complex number, there is at least one complex number such that their product is a real number. $(U = \mathbb{C})$

Problem 3. Which of the following are true? Briefly justify your answer for each. The universe U for each is given in parentheses.

(a) $(\forall x)(x + x \geq x)$ $(U = \mathbb{R})$

(b) $(\forall x)(x + x \geq x)$ $(U = \mathbb{N})$

(c) $(\forall x)(\exists y)(x + y = 0)$ $(U = \mathbb{R})$

(d) $(\exists x)(\forall y)(x + y = 0)$ $(U = \mathbb{R})$

(e) $(\forall y)(\exists x)(\forall z)(xy = xz)$ $(U = \mathbb{R})$

Problem 4. If $P(x)$ is an open sentence, then

(a) $\neg(\forall x)P(x)$ is equivalent to $(\exists x)(\neg P(x))$

(b) $\neg(\exists x)P(x)$ is equivalent to $(\forall x)(\neg P(x))$.

This was stated in class, and Part (a) was proven. Prove Part (b). Rather than include it as an exercise in your manuscript, add it below the existing proof of Part (a).

Problem 5. Let $f(x)$ be a real-valued function. The following sentence is the definition of what it means for $f(x)$ to be continuous at $c \in \mathbb{R}$:

$$(\forall \epsilon > 0)(\exists \delta > 0)\big(|x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon\big).$$

Write the negation of this sentence and simplify it so it does not contain the $\neg$ symbol.