Problem 1. Let $(G, *)$ be a group.

(a) Re-write the definition of a group in terms of quantifiers.

(b) Prove that the identity element is unique.

(c) Prove that each element has a unique inverse.

Problem 2. Let H_1 and H_2 be subgroups of a group G.

(a) Prove that $H_1 \cap H_2$ is a subgroup of G.

(b) Does your argument extend to $\cap_{\alpha \in I} H_\alpha$, where $\{H_\alpha : \alpha \in I\}$ is an arbitrary (not necessarily finite) collection of subgroups? If yes, prove it. If not, give a counterexample.

Problem 3. Let $\phi\colon G \to H$ be a homomorphism, and $K \leq G$ a subgroup. Prove that the *image* of K, denoted $\phi(K) = \{\phi(g) \mid g \in G\}$, is a subgroup of H.

Problem 4. Let $\phi\colon G \to H$ be a homomorphism, and denote the identity elements as e_G and e_H, respectively. Define the *kernel* of ϕ to be the set of elements of G that are sent to the identity in H:

$$\ker(\phi) = \big\{k \in G \mid \phi(k) = e_H\big\}.$$

Prove that $\ker(\phi)$ is a subgroup of G.

Problem 5. Let G be a group. The *center* of G, denoted $Z(G)$, is the set of all elements that commute with everything. That is,

$$Z(G) = \big\{z \in G \mid zg = gz \text{ for all } g \in G\big\}.$$

(a) Find the center of each of the following groups: $\mathbb{Z}$, D_4, and Q_8.

(b) Prove that $Z(G)$ is a subgroup of G.

Problem 6. Verify that the function

$$\phi\colon \mathbb{C} \longrightarrow M_2(\mathbb{R}), \qquad \phi(a+bi) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

is a ring homomorphism. That is, show that for any $z = a + bi$ and $w = c + di$, both $\phi(z + w) = \phi(z) + \phi(w)$ and $\phi(zw) = \phi(z)\phi(w)$ hold.