Problem 1. Recall that a function $f\colon V \to W$ between vector spaces is linear if $f(u+v) = f(u)+f(v)$ and $f(cv) = cf(v)$ hold, for all $u, v \in V$ and $c \in F$. Alternatively, it is linear if

$$f(au+bv) = af(u)+bf(v), \qquad \text{for all } u,v \in V \text{ and } a,b \in F.$$

Prove that these two definitions are equivalent.

Problem 2. Let $f\colon V \to W$ be a linear function between vector spaces.

(a) Prove that $f(\mathbf{0}) = \mathbf{0}$, and that $f(-v) = -f(v)$, for all $v \in V$.

(b) The *nullspace* of f is the set

$$\mathrm{NS}(f) = \{v \in V : f(v) = \mathbf{0}\}.$$

Prove that $\mathrm{NS}(f)$ is a subspace of V.

Problem 3. Prove that the open interval (a, b) in the standard metric $(\mathbb{R}, d)$ is an open set.

Problem 4. Let $A, B \subseteq X$ be open sets in a metric space (X, d). Prove that $A \cup B$ and $A \cap B$ are also open.

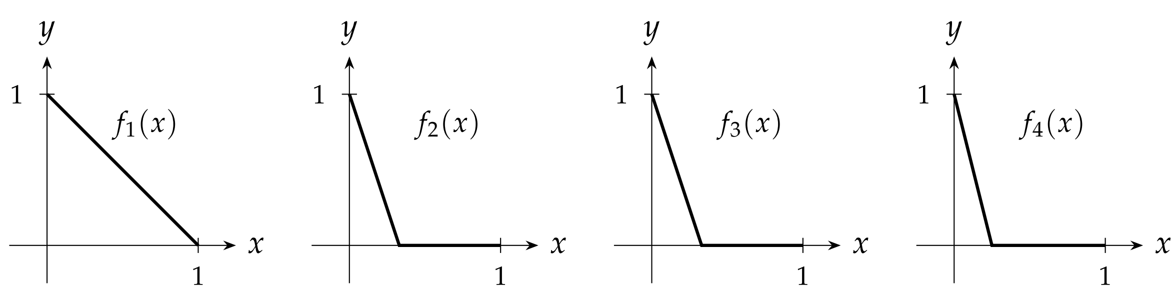
Problem 5. Let $X = \mathcal{C}([0,1])$ be the set of continuous functions on $[0,1]$. Consider the two metric spaces, (X, d_1) and (X, d_∞), where

$$d_1(f,g) = \int_0^1 \big|f(x)-g(x)\big|\,dx, \qquad\qquad d_\infty(f,g) = \max_{x\in[0,1]} \big|f(x)-g(x)\big|.$$

Now, define the sequence of functions $(f_n)_{n\geq 1}$, where

$$f_n(x) = \begin{cases} 1-nx & 0 \leq x < \frac{1}{n}, \\ 0 & \frac{1}{n} < x \leq 1. \end{cases}$$

Picture of the graphs of $f_1, \ldots, f_4$ are shown below.



Answer the following questions for both metric spaces, (X, d_1) and (X, d_∞).

(a) Compute $d(f_n, 0)$, where 0 is the zero function.

(b) Characterize the functions in the unit ball $B_1(0)$ in terms of their graphs.

(c) Determine whether the sequence (f_n) converges or diverges. Prove your answer.

Problem 6. Define the *comb metric* on $X = \mathbb{R}^2$ by the distance function

$$d_{\text{comb}}(x, y) = \begin{cases} |y_1 - y_2| & \text{if } x_1 = x_2, \\ |y_1| + |x_1 - x_2| + |y_2| & \text{if } x_1 \neq x_2. \end{cases}$$

Describe and sketch each of the following open balls in (X, d_{comb}).

1. $B_1\big((0,0)\big)$
2. $B_1\big((0,\frac{1}{2})\big)$
3. $B_1\big((0,1)\big)$.