Problem 1. Let $S \subseteq G$ be a nonempty subset of a group. Define $S^{-1} = \{s^{-1} \mid s \in S\}$. Define the *subgroup generated by* S to be

$$\langle S \rangle = \{s_1 \cdots s_k \mid s_i \in S \cup S^{-1}\}.$$

(a) Prove that $\langle S \rangle$ is actually a subgroup of G.

(b) Prove that
$$\langle S \rangle = \bigcap_{S \subseteq H_\alpha \leq G} H_\alpha.$$
That is, the subgroup generated by S is the intersection of all subgroups of G that contain S.

Problem 2. Let $f\colon X \to Y$ be a function, and $A, B \subseteq X$. Prove the following identities.

(a) $f(A \cap B) \subseteq f(A) \cap f(B)$

(b) $f(A \cup B) = f(A) \cup f(B)$

(c) $f(A \setminus B) \subseteq f(A) \setminus f(B)$.

Problem 3. Prove that the following functions are bijective.

(a) $f\colon \mathbb{R} \to \mathbb{R}$, where $f(x) = x^3$

(b) $f\colon \mathbb{R} \to (0, \infty)$, where $f(x) = e^x$

(c) $f\colon (0, \infty) \to (0, 1)$, where $f(x) = \frac{1}{x+1}$.

Problem 4. Let $f : A \to B$ and $g : B \to C$ be functions.

(a) Show that if $g \circ f$ is surjective then g is surjective.

(b) Show that if $g \circ f$ is injective then f is injective.

(c) Give an example in which $g \circ f$ is surjective, but f is not surjective.

(d) Give an example in which $g \circ f$ is injective, but g is not injective.

Problem 5. Recall that the *nullspace* of a linear map $f\colon V \to W$ between vector spaces is the set of elements that get sent to the zero vector. Similarly, the *kernel* of a homomorphism $\phi\colon G \to H$ between groups is the set of elements that get sent to the identity element:

$$\mathrm{NS}(f) = \{v \in V : f(v) = \mathbf{0}\}, \qquad \ker(\phi) = \{g \in G : \phi(g) = e_H\}.$$

(a) Prove that a linear map $f\colon V \to W$ is injective if and only if $\mathrm{NS}(f) = \{0\}$.

(b) Prove that a group homomorphism $f\colon G \to H$ is injective if and only if $\ker(f) = \{e_H\}$.