Problem 1. Let $f\colon X \to Y$ be a function, and $C, D \subseteq Y$. Prove the following identities.

(a) $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$

(b) $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$

(c) $f^{-1}(C \setminus D) = f^{-1}(C) \setminus f^{-1}(D)$

(d) $f^{-1}(C^c) = (f^{-1}(C))^c$.

Problem 2. Let $f\colon X \to Y$ be a function, and $A \subseteq X$ and $C \subseteq Y$. Prove the following.

(a) $f(f^{-1}(C)) \subseteq C$, with equality iff $C \subseteq \mathrm{im}(f)$.

(b) $A \subseteq f^{-1}(f(A))$, with equality iff f is injective.

Problem 3. Let V and W be vector spaces, and $f\colon V \to W$ a linear function.

(a) Prove that if U is a subspace of V, then $f(U)$ is a subspace of W.

(b) Prove that if Y is a subspace of W, then $f^{-1}(Y)$ is a subspace of V.

Recall that to prove a subset is a subspace, it suffices to show that it contains the zero vector $\mathbf{0}$, and is closed under addition and scalar multiplication.

Problem 4. Let R be the relation on $\mathbb{R}$ defined by

$$a \sim b \quad \Longleftrightarrow \quad |a| = |b|.$$

Prove that this is an equivalence relation, and describe the resulting equivalence classes.

Problem 5. For each of the following relations on $\mathbb{R}^2$, prove that it is an equivalence relation, and then describe the resulting equivalence classes as precisely as you can.

(a) The relation R, defined by

$$(x_1, y_1) \sim (x_2, y_2) \quad \Longleftrightarrow \quad y_1 = y_2.$$

(b) The relation R, defined by

$$(x_1, y_1) \sim (x_2, y_2) \quad \Longleftrightarrow \quad x_1^2 + y_1^2 = x_2^2 + y_2^2.$$

Problem 6. Let $X = \mathcal{C}^1(\mathbb{R})$, the differentiable real-valued functions. Define a relation $\sim$ on X where $f(x) \sim g(x)$ if and only if $f(x)$ and $g(x)$ differ by a constant.

(a) Prove that $\sim$ is an equivalence relation.

(b) Show that $f(x)$ and $g(x)$ are in the same equivalence class if and only if $f'(x) = g'(x)$.