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Coxeter Theory and Discrete Dynamical Systems

A Dissertation submitted in partial satisfaction of the requirement for the degree of Doctor of Philosophy in Mathematics

by

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- Graph dynamical systems (equivalence, morphisms, stochastic systems algorithmic and complexity issues, experimental results);
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- Structure of complex networks, with applications to epidemiology;
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Publications

- **On enumeration of conjugacy classes of Coxeter elements**. (With H. S. Mortveit). *Proceedings of the AMS*. In press. arXiv:0711.1140.

- **Order independence in asynchronous cellular automata**. (With J. McCammond, H. S. Mortveit). *Journal of Cellular Automata*. 3 (1), 2008:37–56. arXiv:0707.2360.

- **Locational market power in power markets**. (With K. Atkins, J. Chen, A. Kumar, A. Marathe). To appear in *Journal of Economic Behavior and Organization*, 2008.

- **Cycle equivalence of graph dynamical systems**. (With H. S. Mortveit). Submitted to *Physica D*, 2007. arXiv:0802.4412.

- **Equivalences on acyclic orientations**. (With H. S. Mortveit). Submitted to *Discrete Mathematics*, 2008. arXiv:0709.0291.

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- **Locational market power in power markets**. (With K. Atkins, J. Chen, A. Kumar, A. Marathe). *Proceedings of the 29th IAEE International Conference*. Potsdam, Germany. June 2006.

- **The mathematics of juggling** (Book review). *Math Horizons*, February 2004.

- **Thinking outside the box, and over the elephant**. (With M. Banister, M. Smukler). *The Journal of Undergraduate Mathematics and Its Applications.* 24 (3), 2003.

- **Applications of Rank Functions of Graphs**. In preparation.

- **Update Order Instability in Graph Dynamical Systems**. (With V.S.A. Kumar, H. S. Mortveit). In preparation.

- **Dynamics groups of asynchronous cellular automata**. (With J. McCammond, H. S. Mortveit). In preparation.

- **A mathematical and computational study of constraint induced market power**. (With J. Chen, V.S.A. Kumar, A. Marathe). In preparation.

- **Automorphism groups of stochastic sequential dynamical systems**. In preparation.

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Abstract

Coxeter Theory and Discrete Dynamical Systems

by Matthew Macauley

This dissertation is the first study to apply the theory of Coxeter groups to a class of discrete dynamical systems called *sequential dynamical systems*, or "SDSs," which were first invented in the late 1990s. An SDS is defined over a graph Y, where the vertices take on one of several states from a finite set K, and there is a sequence of Y-local functions $\mathfrak{F}_Y = (F_i)_{i\in\mathrm{v}[Y]}$, which are applied sequentially to obtain the global update function. After a brief introduction, we begin by examining several notions of equivalence, where the choice of functions is fixed but the update order can potentially vary. Each of these gives rise to a corresponding combinatorial structure called a neutral network which encapsulates the key information about the sensitivity of the features of the dynamics to changes in the update order. These networks and their properties have helped unravel new connections between SDSs and diverse fields such as hyperplane arrangements, permutahedra, and Coxeter groups. In particular, we use existing results on Coxeter groups to prove theorems about the dynamics of SDSs, as well as extend current results about Coxeter theory. One of the main theorems is a recurrence relation for the number of conjugacy classes of Coxeter elements, via edge deletion and contraction of non-bridge edges of the Coxeter graph, and this has a direct bearing on the number of SDS maps up to cycle equivalence. We show how the local functions of an SDS generate the dynamics group, which is a homomorphic image of a Coxeter group, and this is insightful for the class of SDSs that are word-independent. Upon examination of the 256 elementary asynchronous cellular automata (ACAs), we prove that exactly 104 of them are word-independent. This is a significant extension of a recent result that stated the same result for 11 out of the 16 symmetric ACAs. To better understand these systems, We study their periodic point sets and dynamics groups. Finally, we consider SDSs where the base graph and update order are allowed to have a certain degree of stochasticity. We show how symmetries in the distribution of these constituents appear as symmetries in the corresponding phase spaces, or Markov chains. Finally, we discuss update order instability of stochastic SDSs, and give examples of how various notions of instability measures that arise from previous work in this dissertation are dependent on the sparsity of the base graph, sometimes in completely opposite ways.

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1. Introduction

1.1. **History and motivation.** The focus of this dissertation is on a class of finite discrete dynamical systems called *sequential dynamical systems,* or "SDSs." An SDS consists of a graph Y where each vertex v takes on a state x_v from a finite set K, with a sequence of functions that determine a how to update a state based on the states of its neighbors. The SDS map is constructed by composing these functions in a particular order, and it is the iteration of this map that determines the dynamics of the system, which is described by the phase space. SDSs are part of a more general class of finite discrete dynamical systems, called *graph dynamical systems*, where there are fewer restrictions on how the states are updated. One classic example of a graph dynamical system is a finite *cellular automaton*, or "CA." A finite CA is defined much like an SDS, but the underlying graph is a regular grid, such as a circular graph, and the functions are updated in parallel, rather than sequentially. SDSs were invented to help model complex systems where the order that the update rules are applied plays a significant role in the global dynamics. Moreover, they are attractive mathematical objects to study in their own right, and have a number of connections to other areas of mathematics. Cellular automata are typically regarded as the first type of graph dynamical system to be studied, invented by Stanisław Ulam and John von Neumann, while working at Los Alamos National Laboratory in the 1940s [40]. Their primary motivation was to use them as models of biological systems [45]. In 1969, German computer scientist Konrad Zuse proposed that the universe is essentially one big cellular automaton [47].

In the 1970s, John Conway invented the Game of Life, a two-dimensional CA, which was later popularized by Martin Gardner [16]. Beginning in 1983, Stephen Wolfram published a series of papers devoted to developing a theory of CAs and their role in science [28, 42, 43, 44]. This is a central theme in Wolfram's 1280-page book *A New Kind of Science*, published in 2002.

The idea of a sequential dynamical system did not come to life until the late 1990s, when the were invented by a group of scientists and mathematicians, also at Los Alamos. They were part of the Simulation Science Group at the laboratory, and had a practical need to build a generic mathematical framework for computer simulations of large-scale agent-based socio-technological systems [6, 7, 8, 9]. In such settings, agents are represented as vertices in a network, and edges correspond to some type of communication or contact between agents. Agents typically are in one of several states, and the evolution of an agent's state is a function of the states of its neighbors. Together, the interaction of these individual agents make up the global complex system. The goal became to mathematically and computationally capture the key properties of such a generic setting, and build an underlying framework that could be used in diverse settings, from epidemiological models to transportation networks. Thus was born the concept of a sequential dynamical system.

Many natural questions about SDSs immediately arose, such as how sensitive the dynamics are to variations in the update order, or what kind of effects a small change in the underlying graph, such as adding or removing an edge, can have on the global dynamics of the system. In the past decade, much work has been done to develop the fundamentals of SDSs [6, 7, 8, 9, 30]. There has also been a significant effort to understand the computational, complexity and algorithmic issues involved [1, 2, 3, 4, 5]. Recently, promising connections have been made between SDSs and diverse fields such as Coxeter groups. We develop these ideas throughout this dissertation, as well as explore another relativity new area, that of stochastic sequential dynamical systems.

1.2. **Preliminaries.** To begin, we will define some terms and state some notational conventions that will hold throughout this dissertation. Let Y be an undirected graph with vertex set $\mathrm{v}[Y] = \{1, \ldots, n\}$ and edge set $\mathrm{e}[Y]$. Each vertex in Y has a *vertex state* from a finite set K, and so the set of all states is K^n. In most of the SDS literature, $K = \mathbb{F}_2 = \{0, 1\}$, because this is common in applications and models. However, we shall not make this assumption in this document unless we explicitly state otherwise. Throughout, $\mathbf{y}$ will denote a state in K^n, and $y_i = \mathsf{proj}_i(\mathbf{y}) \in K$ will denote the state of vertex i. Denote the 1-neighborhood of vertex i by $N_{1,Y}(i) := \{j \in \mathrm{v}[Y] : \{i, j\} \in \mathrm{e}[Y]\} \cup \{i\}$, and set $d_i = |N_{1,Y}(i)| - 1$, the degree of vertex i. For any $\mathbf{y} \in K^n$, let $\mathbf{y}[i]$ denote the restriction of $\mathbf{y}$ to coordinates in $N_{1,Y}(i)$.

Definition 1.1 (Vertex function)**.** For any $i \in \mathrm{v}[Y]$, a *vertex function* of vertex i is a function $f_{i,Y} \colon K^{d_i+1} \to K$ where the domain is the set of states of the vertices in $N_{1,Y}(i)$.

Definition 1.2 (Local function)**.** For any $i \in \mathrm{v}[Y]$, a *local function* of vertex i is a function $F_{i,Y} \colon K^n \to K^n$ such that (i) $F_{i,Y}$ only changes the

i^{th} coordinate of $\mathbf{y} \in K^n$, and (ii) If $\mathbf{y}$ and $\mathbf{y}'$ agree when restricted to the coordinates in $N_{1,Y}(1)$, then $F_{i,Y}(\mathbf{y}) = F_{i,Y}(\mathbf{y}')$.

It is clear from the definition that every Y-local function $F_{i,Y}$ is determined by the extension of a unique vertex function $f_{i,Y}$, by

$$F_{i,Y}(y_1, \ldots, y_n) = (y_1, \ldots, y_{i-1}, f_{i,Y}(\mathbf{y}[i]), y_{i+1}, \ldots, y_n) \quad . \tag{1.1}$$

and vice-versa. We denote a sequence of vertex functions by $\mathfrak{f}_Y = (f_{i,Y})_{i=1}^n$ and the corresponding local functions by $\mathfrak{F}_Y = (F_{i,Y})_{i=1}^n$. The graph Y is called the *base graph*, or *dependency graph.* If Y is clear from the context, we will omit it from the subscript of the individual functions.

Definition 1.3 (Symmetric and homogeneous functions)**.** Let $\mathfrak{f}_Y = (f_{i,Y})_{i=1}^n$ be a sequence of vertex functions on Y. Then $\mathfrak{f}_Y$ is said to be *symmetric* if for each i,

$$f_{i,Y}(y_{i_1}, \ldots, y_{i_k}) = f_{i,Y}(y_{\pi(i_1)}, \ldots, y_{\pi(i_k)}) \ , \tag{1.2}$$

for all $\pi \in S_k$. Moreover, $\mathfrak{f}_Y$ is *quasi-symmetric* if (1.2) holds for all $\pi \in S_k$ that fix the i^{th} coordinate. A sequence of symmetric functions $\mathfrak{f}_Y$ is said to be *homogeneous* if $|N_{1,Y}(i)| = |N_{1,Y}(j)|$ implies that $f_{i,Y} = f_{j,Y}$. Finally, we say that the corresponding sequence $\mathfrak{F}_Y = (F_{i,Y})_{i=1}^n$ of Y-local functions is symmetric (or homogeneous) if $\mathfrak{f}_Y$ is symmetric (or homogeneous).

Most of the SDS literature has the blanket assumption that the local functions are homogeneous and symmetric. This is generally a much stronger statement than is required. Many of the existing theorems hold as long as the functions are invariant under the automorphism group of the base graph Y. We define this more generally because sometimes the functions are not invariant under the entire automorphism group of Y, but rather just a subgroup.

Definition 1.4 (G-invariant functions)**.** If G is a subgroup of $\mathsf{Aut}(Y)$, then a sequence of Y-local functions $\mathfrak{F}_Y$ is *G-invariant* if $\varphi \circ F_i = F_{\varphi(i)} \circ \varphi$ for every $i \in [n]$ and $\varphi \in G$. A sequence of vertex functions $\mathfrak{f}_Y$ is G-invariant if the corresponding sequence $\mathfrak{F}_Y$ of Y-local functions is G-invariant.

We will not go back and reprove every existing SDS result with this weaker condition, but state that $\mathsf{Aut}(Y)$-invariance is precisely the property that is being used, which trivially holds for symmetric homogeneous functions. We will later see this explicitly for one of the central existing results.

Definition 1.5 (Update orders)**.** Let W_Y be the set of all words over $\mathrm{v}[Y]$, and let S_Y be the subset of all total orderings of $\mathrm{v}[Y]$, i.e., words where every vertex appears precisely once. A word ω of length $|\omega| = m$ will be denoted $\omega_1\omega_2 \cdots \omega_m$, $(\omega_1, \omega, \ldots, \omega_m)$, $(\omega(1), \omega(2), \ldots, \omega(m))$, etc. We will refer to elements of S_Y as *simple update orders*, or permutations, because they can be canonically associated with permutations of $\mathrm{v}[Y]$. Typically we will denote words by ω and ζ, but when speaking specifically about simple update orders, we will use π and σ. Finally, a word $\omega \in W_Y$ is *fair* if for every $i \in \mathrm{v}[Y]$, $\pi_j = i$ for some $1 \leq j \leq |\omega|$.

Definition 1.6 (Sequential dynamical system)**.** A *sequential dynamical system*, or "SDS", is a triple $(Y, \mathfrak{F}_Y, \omega)$ consisting of an undirected graph Y, a sequence of Y-local functions $\mathfrak{F}_Y$, and an update order $\omega \in W_Y$, say of length $|\omega| = m$. The function defined by

$$[\mathfrak{F}_Y, \omega] \colon K^n \longrightarrow K^n, \qquad [\mathfrak{F}_Y, \omega] = F_{\omega_m} \circ \cdots \circ F_{\omega_1} \ .$$

is the SDS *map*. If $\omega \in S_Y$, then $(Y, \mathfrak{F}_Y, \omega)$ is a *permutation* SDS.

The defining characteristic of any dynamical system is its phase space, which represents every possible state of the system, and transitions between them. For continuous systems, the dynamics is governed by a vector field. For a finite dynamical system, or any function from a finite set to itself, the phase space is simply a directed graph.

Definition 1.7 (Phase space)**.** For a finite dynamical system $\phi \colon K^n \to K^n$, the *phase space*, denoted $\Gamma(\phi)$, is the directed graph with

$$\mathrm{v}[\Gamma(\phi)] = K^n, \qquad\qquad \mathrm{e}[\Gamma(\phi)] = \{(\mathbf{y}, \phi(\mathbf{y})) \mid \mathbf{y} \in K^n\} \ .$$

1.3. **Group actions.** There are several standard group actions that will frequently appear throughout this dissertation. The group S_n acts on K^n by

$$\sigma \cdot \mathbf{y} = \left(y_{\sigma^{-1}(1)}, \ldots, y_{\sigma^{-1}(n)}\right) \ . \tag{1.3}$$

and on W_Y by

$$\sigma * \omega = (\sigma(\omega_1), \ldots, \sigma(\omega_n)) \ , \tag{1.4}$$

The difference here is that in (1.3), the group element is permuting the indices of the n-tuple, whereas in (1.4), it is permuting the values of the entries. Frequently, we will be composing a function $\Phi\colon K^n \to K^n$ with an automorphism $\varphi \in \mathsf{Aut}(Y) \leq S_n$. This will be written $\Phi \circ \varphi$ (or $\varphi \circ \Phi$), and it is understood that φ is permuting the coordinates of K^n as in (1.3). Explicitly, this is

$$(\Phi \circ \varphi)(\mathbf{y}) = \Phi(\varphi \cdot \mathbf{y}) \ , \qquad (\varphi \circ \Phi)(\mathbf{y}) = \varphi \cdot (\Phi(\mathbf{y})) \ .$$

2. Equivalence of SDSs

In the next two sections, we will discuss three different notions of equivalence between finite dynamical systems, and how they apply in the setting of SDSs when the sequence of functions is fixed and the update order is varied. Two maps $\phi, \psi\colon K^n \to K^n$ are *functionally equivalent* if they are identical as functions, *dynamically equivalent* if their phase spaces are isomorphic as digraphs, and *cycle equivalent* if their phase spaces restricted to the periodic points are isomorphic as digraphs. For each of these notions, we present theorems that describe when two SDS maps $[\mathfrak{F}_Y, \pi]$ and $[\mathfrak{F}_Y, \sigma]$ are equivalent. This leads to natural equivalence relations on the set S_Y of all simple update orders. We will study these by defining a graph called a *neutral network*, where the vertices correspond with collections of update orders, and two vertices on the same connected component give rise to equivalent SDS maps. Thus, the stability of an SDS map is encoded in the structure of the corresponding neutral network. The number of components is a complexity measure of the system, because counting these equivalence classes gives an upper bound for the number of non-equivalent SDSs obtainable by varying the update order. We study these combinatorial bounds and discuss their sharpness.

In this section, we focus on functional and dynamical equivalence. The concepts have appeared in the literature [30], but we present a new construction for the neutral network for functional equivalence using hyperplane arrangements. This reproves an old result, and provides a connection to the rank function of a graph, which has both a geometric and a combinatorial interpretation. The concept of cycle equivalence is relegated to Section 3, where we demonstrate a connection to fields such as Coxeter theory, quiver representations, and graph polynomials. We also further explore and develop the underlying discrete mathematics.

2.1. **Functional equivalence.**

2.1.1. *Acyclic orientations and update graphs.* Two SDSs are *functionally equivalent* if their SDS maps are identical as functions. For a fixed sequence of functions $\mathfrak{F}_Y$, a natural question to ask is when does $[\mathfrak{F}_Y, \omega] = [\mathfrak{F}_Y, \zeta]$ for $\omega, \zeta \in W_Y$? The *word update graph* $\hat{U}(Y)$ provides a partial answer to this. The vertex set of $\hat{U}(Y)$ is W_Y, and two words $\omega \neq \zeta$ of length m are adjacent if they differ by a single transposition of two adjacent entries ω_i and ω_{i+1} such that $\{\omega_i, \omega_{i+1}\} \notin \mathrm{e}[Y]$. The finite subgraph $U(Y)$ of $\hat{U}(Y)$ induced by the vertex set S_Y is called the *permutation update graph*, or just "update graph" and is denoted $U(Y)$. Clearly, it is a union of connected components of $\hat{U}(Y)$.

Example 2.1. Let Circ_n be the graph with vertex set $\mathrm{v}[\mathsf{Circ}_n] = \{1, \ldots, n\}$ and edges $\{i, i+1\}$ modulo n. The permutation update graph $U(\mathsf{Circ}_4)$ has 14 connected components as shown in Figure 2.1.

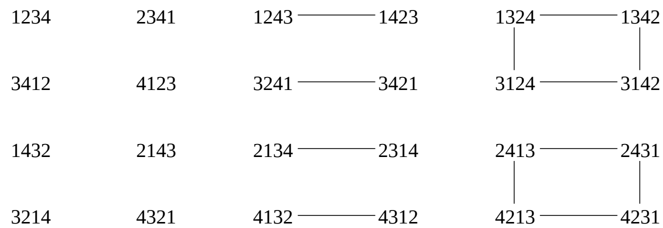


FIGURE 2.1. The update graph $U(\mathsf{Circ}_4)$.

For now, we will discuss $U(Y)$ rather than $\hat{U}(Y)$. The same concepts that hold for permutation update orders carry over to general word update orders under slight modifications, but the notation is more cumbersome and no additional significant insight is gained. Let $\sim_Y$ be the equivalence relation on S_Y defined by $\pi \sim_Y \sigma$ if π and σ belong to the same connected component of $U(Y)$. We denote the equivalence class of S_Y containing π as $[\pi]_Y$, and the set of equivalence classes by $S_Y/\sim_Y$. By construction, $\pi \sim_Y \sigma$ implies the equality $[\mathfrak{F}_Y, \pi] = [\mathfrak{F}_Y, \sigma]$. It follows that for a fixed sequence of functions $\mathfrak{F}_Y$, $|S_Y/\sim_Y|$ is an upper bound for the number of SDS maps $[\mathfrak{F}_Y, \pi]$ up to functional equivalence, where $\pi \in S_Y$. The next result discusses the sharpness of this bound.

Proposition 2.2 ([29])**.** *If $\mathfrak{f}_Y$ are the boolean* nor *functions, defined by*

$$\mathsf{nor}_k \colon \mathbb{F}_2^k \longrightarrow \mathbb{F}_2, \qquad \mathsf{nor}_k(y_1, \ldots, y_k) = \prod_{i=1}^{k} (1 + y_i) \ ,$$

and Nor_Y *the corresponding* Y*-local functions, then* $[\mathsf{Nor}_Y, \pi] = [\mathsf{Nor}_Y, \sigma]$ *holds if and only if* $\pi \sim_Y \sigma$.

Proposition 2.2 shows that for any graph Y, there exists a sequence of Y-local functions such that $|S_Y/{\sim_Y}|$ is a sharp upper bound for the number of functionally non-equivalent permutation SDS maps obtainable by varying the update order.

An alternative way to characterize $S_Y/{\sim_Y}$, and functional equivalence of SDSs, is through acyclic orientations of the base graph. Orientations of Y are represented as maps $O_Y \colon \mathrm{e}[Y] \longrightarrow \mathrm{v}[Y] \times \mathrm{v}[Y]$, which may be viewed as directed graphs. The set of acyclic orientations of Y is denoted $\mathsf{Acyc}(Y)$, and we set $\alpha(Y) = |\mathsf{Acyc}(Y)|$. Every acyclic orientation defines a partial ordering on $\mathrm{v}[Y]$ where the covering relations are $i \leq_{O_Y} j$ if $\{i,j\} \in \mathrm{e}[Y]$ is oriented $O_Y(\{i,j\}) = (i,j)$. This allows us to refer to an orientation O_Y as a poset. The set of linear extensions of O_Y contains precisely the simple update orders $\pi \in S_Y$ such that if $i \leq_{O_Y} j$, then i precedes j in π. Therefore, every permutation $\pi \in S_Y$ induces a canonical linear order of $\mathrm{v}[Y]$, and hence an acyclic orientation $O_Y^\pi \in \mathsf{Acyc}(Y)$, where $O_Y^\pi(\{i,j\}) = (i,j)$ if $i \leq_\pi j$ and $O_Y^\pi(\{i,j\}) = (j,i)$ otherwise. The bijection

$$f_Y \colon S_Y/{\sim_Y} \longrightarrow \mathsf{Acyc}(Y)\,, \qquad f_Y([\pi]_Y) = O_Y^\pi\,, \tag{2.1}$$

from [34] lets us interpret an equivalence classes $[\pi]_Y$ as an acyclic orientation. The number of equivalence classes under $\sim_Y$ is therefore given by $\alpha(Y)$.

For $O_Y \in \mathsf{Acyc}(Y)$ and $e = \{v, w\} \in \mathrm{e}[Y]$, let $O_Y^{\rho(e)}$ be the orientation of Y obtained from O_Y by reversing the edge-orientation of e. Let Y'_e and Y''_e denote the graphs obtained from Y by deletion and contraction of e, respectively, and let $O_{Y'_e}$ and $O_{Y''_e}$ be the resulting orientations achieved from O_Y under these operations. The bijection

$$\mathsf{Acyc}(Y) \longrightarrow \mathsf{Acyc}(Y'_e) \cup \mathsf{Acyc}(Y''_e) \tag{2.2}$$

defined by

$$O_Y \quad \longmapsto \quad \begin{cases} O_{Y'_e} & O_Y^{\rho(e)} \notin \mathsf{Acyc}(Y), \\ O_{Y'_e} & O_Y^{\rho(e)} \in \mathsf{Acyc}(Y) \text{ and } O_Y(e) = (v, w), \\ O_{Y''_e} & O_Y^{\rho(e)} \in \mathsf{Acyc}(Y) \text{ and } O_Y(e) = (w, v). \end{cases} \tag{2.3}$$

is well-known, and shows that the quantity $\alpha(Y)$ can be computed through the recursion relation

$$\alpha(Y) = \alpha(Y'_e) + \alpha(Y''_e)\ , \tag{2.4}$$

and this is holds for any $e \in \mathrm{e}[Y]$.

It follows immediately from (2.1) that for a fixed choice of Y-local functions $\mathfrak{F}_Y$, the quantity $\alpha(Y)$ is an upper bound for the number permutation SDS maps $[\mathfrak{F}_Y, \pi]$ up to functional equivalence. By Proposition 2.2, this bound can be sharp. As alluded to earlier, these results can be extended to general word update orders $\omega \in W_Y$. We refer the interested reader to [35].

2.1.2. *Permutahedra.* At this time, we present an alternative construction of the update graph $U(Y)$, by cutting the 1-skeleton of a permutahedron with hyperplanes through its center.

Definition 2.3 (Permutahedron)**.** The *n-permutahedron*, denoted Π_n, is the convex hull of all permutations of the n-tuple $(1, \dots, n) \in \mathbb{R}^n$.

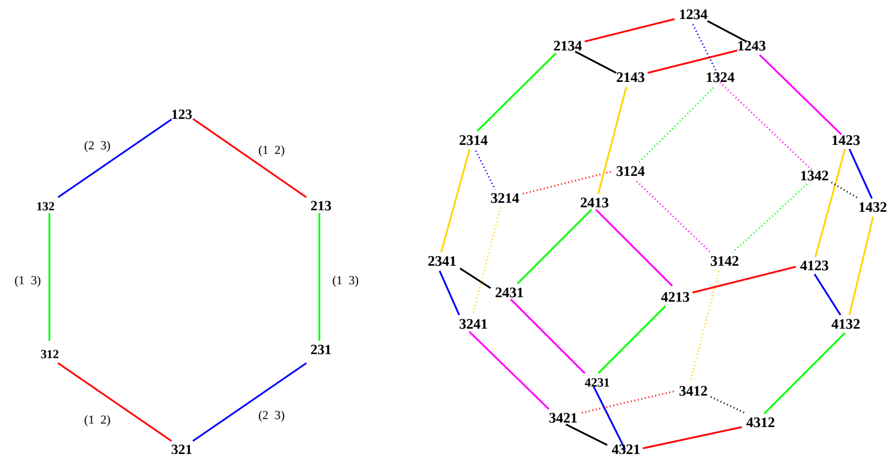
The n-permutahedron is an $(n-1)$-dimensional polytope, because embedded in $\mathbb{R}^n$, the vertices of Π_n all lie on the hyperplane $\sum_{i=1}^{n} x_i = \frac{n(n-1)}{2}$. The n-permutahedron has some very nice properties: it is vertex-transitive (under the action of S_n), each edge has length $\sqrt{2}$, and two vertices are adjacent iff they differ by swapping two coordinates with adjacent values. This gives rise to the *geometric labeling* of the vertices and edges of Π_n:

- The vertices of Π_n are labeled such that two vertices are adjacent if they differ by swapping two coordinates with consecutive *values,*
- An edge of Π_n is labeled with the transposition $(x_i \;\; x_j)$ of the values of the two entries that are swapped.

Alternatively, the vertices and edges of Π_n can be labeled as follows [46]:

- The vertices of Π_n are labeled such that two vertices are adjacent if they differ by swapping two coordinates in consecutive *positions.*
- An edge of Π_n is labeled with a transposition $(x_i \;\; x_j)$ of the values of the two entries that are swapped.

We call this the *combinatorial labeling* of Π_n, and this is the labeling that we will work with primarily. The polytopes Π_3 and Π_4, with this labeling, are shown in Figure 2.2. The edge labels are omitted in Π_4, but it should be clear what they are. We use the combinatorial labeling of Π_n because of



(a) Π_3 (b) Π_4

FIGURE 2.2. Permutahedra, for $n = 3$ and $n = 4$, with the combinatorial labeling. The edge labels are omitted in Π_4.

several key properties: (i) its 1-skeleton is the update graph of E_n, the graph on n vertices and no edges, and (ii) every transposition in S_n corresponds to a complete set of parallel edges of Π_n. From these observations, we get a nice method for constructing the update graph $U(Y)$. Let $H^n_{i,j}$ be the hyperplane through the center of Π_n, normal to the parallel edges with label $(i\ \ j)$, and let $v^n_{i,j}$ be the normal vector to $H^n_{i,j}$ (orientation is irrelevant).

Proposition 2.4. *The update graph $U(Y)$ can be constructed by "cutting" the 1-skeleton of Π_n by $H^n_{i,j}$ for each $\{i,j\} \in \mathrm{e}[Y]$, i.e., removing all edges of Π_n that intersect $H^n_{i,j}$.*

Let $\mathcal{H}(Y)$ be the hyperplane arrangement $\{H^n_{i,j} \mid \{i,j\} \in \mathrm{e}[Y]\}$, and $\mathcal{C}(\mathcal{H}(Y))$ be the set of chambers. It follows immediately that

$$|S_Y/{\sim_Y}| = |\mathcal{C}(\mathcal{H}(Y))| \,. \tag{2.5}$$

From (2.1) and (2.5), we know that $|\mathcal{C}(\mathcal{H}(Y))| = \alpha(Y)$. However, this can be computed directly, by recognizing that $\mathcal{H}(Y)$ is a in fact a special arrangement known as the graphic arrangement of Y.

Definition 2.5 (Graphic arrangement)**.** Let $f_1, \ldots, f_n$ be a basis for the dual space of $\mathbb{R}^n$. For a graph Y, let $\mathcal{H}(Y)$ be the arrangement defined by

$$\mathcal{H}(Y) = \{\ker(f_i - f_j) \mid \{i, j\} \in \mathrm{e}[Y]\} .$$

$\mathcal{H}(Y)$ is called the *graphic arrangement* of Y.

From the geometric labeling of Π_n, it is easy to see that a normal vector to the hyperplane $H^n_{i,j}$ is $v^n_{i,j} = \mathbf{e}_i - \mathbf{e}_j$. Thus, using the standard basis of the dual space of $\mathbb{R}^n$, the hyperplanes in the graphic arrangement of Y are precisely the hyperplanes $\{H^n_{i,j}\}$ constructed to create the update graph $U(Y)$ from Π_n. The bijection between the components $\mathcal{C}(\mathcal{H}(Y))$ of the graphic arrangement of Y, and $\mathsf{Acyc}(Y)$, is well-known [33]. In summary, we have bijections between the following sets.

$$\mathcal{C}(\mathcal{H}(Y)) \quad \longleftrightarrow \quad \mathsf{Acyc}(Y) \quad \longleftrightarrow \quad S_Y/{\sim_Y} \quad \longleftrightarrow \quad \{[\mathsf{Nor}_Y, \pi] \mid \pi \in S_Y\} .$$

Individually, none of these bijections are new, but the construction of the update graph from the graphic hyperplane arrangement provides us with a new way to link them.

2.1.3. *Rank functions.* We now pause to analyze $U(Y)$ and $\alpha(Y)$ for a few special cases. It is obvious that if Y is a tree with m edges, then $\alpha(Y) = 2^m$.

This is equivalent to the vectors $\{v^n_{i,j} \mid \{i,j\} \in \mathrm{e}[Y]\}$ being linearly independent. If $Y = \mathsf{Circ}_n$, then $\alpha(Y) = 2^n - 2$. To illustrate the construction of $U(Y)$ by hyperplane cuts, we will consider an explicit example. If Y has 4 vertices, then $U(Y)$ is constructed from Π_4, as in Figure 2.2. Notice that if Y contains a 3-cycle, say $1 \to 2 \to 3 \to 1$, then this corresponds to three hyperplanes that each cut Π_4 (with the combinatorial labeling) through a pair of antipodal hexagonal faces with edge labels (1 2), (2 3), and (3 1), as shown in Figure 2.3. The vectors $\{v^3_{1,2}, v^3_{1,3}, v^3_{2,3}\}$ form a minimal linearly

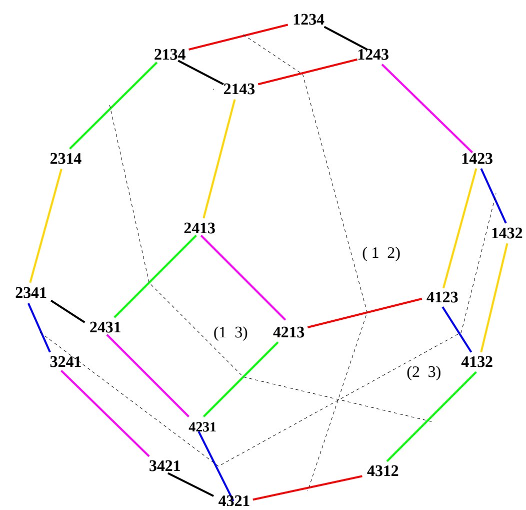


FIGURE 2.3. Hyperplanes cuts corresponding with the edges $\{1,2\}$, $\{2,3\}$, and $\{1,3\}$ in $Y < K_4$.

dependent set. This generalizes as follows. A $2d$-gon facet of Π_n corresponds to a length-d cycle in Y, and a set of d vectors $\{v^n_{i,j}\}$ that span a $(d-1)$-dimensional subspace. Removing any one edge of the length-d cycle makes the resulting subgraph a tree, and the corresponding $d-1$ vectors $\{v^n_{i,j}\}$ are linearly independent. Thus, cycles in Y correspond with minimal linearly dependent sets of vectors. Since the hyperplane arrangement $\mathcal{H}(Y)$

associates a graph with a set of vectors, it is well-founded to define the rank function of a graph as follows.

Definition 2.6 (Rank function of a graph)**.** Let Y be an undirected graph on n vertices. For every edge $\{i,j\}$, let $H^n_{i,j}$ be the corresponding hyperplane of $\mathcal{H}(Y)$ with normal vector $v^n_{i,j}$. If $\mathcal{P}(Y)$ is the set of all subgraphs of Y with vertex set $\mathrm{v}[Y]$ (i.e., formed by removing edges), then define the *rank function* of Y by

$$r_Y\colon \mathcal{P}(Y) \longrightarrow \mathbb{Z}\,, \qquad\qquad r_Y(Z) = \dim\langle v^n_{i,j} \mid \{i,j\} \in \mathrm{e}[Z]\rangle\,,$$

This definition of the rank function of a graph is geometric, based upon a set of Euclidean hyperplanes from the graphic arrangement. This makes it potentially difficult to compute for a given graph. However, the following proposition describes it in a purely combinatorial setting.

Proposition 2.7. *If Z is a subgraph of Y with $\mathrm{v}[Y] = \mathrm{v}[Z]$, then*

$$r_Y(Z) = |\mathrm{v}[Y]| - n(Z)\,, \tag{2.6}$$

where $n(Z)$ is the number of connected components of Z.

Proof. We claim that $r_Y(Z)$ is precisely the number of edges in a spanning forest of Z. To see this, consider a spanning forest F with m edges. The corresponding vectors $\{v^n_{i',j'} \mid \{i',j'\} \in \mathrm{e}[F]\}$ form a linearly independent set, thus $r_Y(Z) \geq m$. Moreover, since it is impossible to add an edge to a spanning forest without introducing a cycle, every edge (vector) $\{i,j\} \in$

$\mathrm{e}[Z] \setminus \mathrm{e}[F]$ is a linear combination of edges (vectors) in $\mathrm{e}[F]$. Therefore, $r_Y(Z) \leq m$, and we now conclude that $r_Y(Z) \leq m$.

If Z has connected components $\mathcal{C}_1, \ldots C_k$, then any spanning tree of $\mathcal{C}_i$ has $|\mathrm{v}[\mathcal{C}_i]| - 1$ edges. The number of edges in the spanning forest is thus

$$|\mathrm{e}[F]| = \sum_{i=1}^{k} (|\mathrm{v}[\mathcal{C}_i]| - 1) = |\mathrm{v}[Y]| - n(Z) \ ,$$

and this completes the proof. $\square$

We comment that the rank function can be defined for any matroid [13]. However, this is unnecessary in this context, and obscures the key geometric intuition.

2.2. **Dynamical equivalence.** The next type of equivalence that we will discuss is dynamical equivalence, which is weaker than functional equivalence.

Definition 2.8 (Dynamical equivalence)**.** Two finite dynamical systems $\phi, \psi \colon K^n \to K^n$ are *dynamically equivalent* if there is a bijection $h \colon K^n \to K^n$ such that $\phi \circ h = h \circ \psi$.

This definition is equivalent to the phase spaces $\Gamma(\phi)$ and $\Gamma(\psi)$ being isomorphic as digraphs. Over the discrete topology on K^n, the concepts of dynamical equivalence and topological conjugation coincide. As with functional equivalence, it is natural to ask how many SDS maps can be obtained up to dynamical equivalence, given a fixed sequence of functions $\mathfrak{F}_Y$, by varying the update order. In general, this is not known, although an upper bound $\bar{\alpha}(Y)$ can be derived. First, the following proposition must be established.

Proposition 2.9 ([29]). *If $\mathfrak{F}_Y$ is a sequence of $\mathsf{Aut}(Y)$-invariant Y-local functions, then*

$$[\mathfrak{F}_Y, \varphi * \pi] = \varphi \circ [\mathfrak{F}_Y, \pi] \circ \varphi^{-1} \tag{2.7}$$

for all $\pi \in S_Y$ and all $\varphi \in \mathsf{Aut}(Y)$.

We note that in [29], Proposition 2.9 is stated for the stronger condition of $\mathfrak{F}_Y$ being homogeneous and symmetric. However, these extra conditions are unnecessary. As long as $\mathfrak{F}_Y$ is $\mathsf{Aut}(Y)$-invariant, the diagram

$$\begin{tikzcd}
K^n \arrow[r, "F_{\pi(n)}"] \arrow[d, "\varphi"'] & K^n \arrow[r, "F_{\pi(n-1)}"] \arrow[d, "\varphi"] & \cdots \arrow[r, "F_{\pi(1)}"] & K^n \arrow[d, "\varphi"] \\
K^m \arrow[r, "F_{\varphi(\pi(n))}"] & K^m \arrow[r, "F_{\varphi(\pi(n-1))}"] & \cdots \arrow[r, "F_{\varphi(\pi(2))}"] & K^m
\end{tikzcd} \tag{2.8}$$

commutes, and the bottom row is easily seen to be both sides of the equation in (2.7). As a consequence of Proposition 2.9, the group $\mathsf{Aut}(Y)$ acts on $S_Y/{\sim_Y}$ by $\varphi \cdot [\pi]_Y = [\varphi * \pi]_Y$. It follows that the number orbits in $S_Y/{\sim_Y}$ under this action is an upper bound for the number of permutation SDS maps $[\mathfrak{F}_Y, \pi]$ up to dynamical equivalence, for a fixed choice of $\mathfrak{F}_Y$. We denote this quantity by $\bar{\alpha}(Y)$. For each $\varphi \in \mathsf{Aut}(Y)$, let $\langle\varphi\rangle \setminus Y$ denote the *orbit graph* of the cyclic group $G = \langle\varphi\rangle$ and Y: the vertices are the orbits of the action of $\langle\varphi\rangle$ on $\mathrm{v}[Y]$, and edges are respected. An application of Burnside's Lemma, and a careful analysis of the fixed point sets of the action, yields

$$\bar{\alpha}(Y) = \frac{1}{|\mathsf{Aut}(Y)|} \sum_{\varphi \in \mathsf{Aut}(Y)} |\mathsf{Fix}(\varphi)| = \frac{1}{|\mathsf{Aut}(Y)|} \sum_{\varphi \in \mathsf{Aut}(Y)} \alpha(\langle\varphi\rangle \setminus Y) \,. \tag{2.9}$$

The details of this can be found in [7, 8]. The bound $\bar{\alpha}$ is known to be sharp for certain graph classes [8], but the sharpness question in the general case is still an open problem. It is conjectured that as with functional equivalence, SDS maps with $\mathfrak{F}_Y = \mathsf{Nor}_Y$ achieve this upper bound.

3. Cycle Equivalence and Coxeter Theory

The weakest form of equivalence that we shall discuss is cycle equivalence. While functional and dynamical equivalence have been studied in the context of SDSs, cycle equivalence has not. The concept appeared in a paper on finite dynamical systems, under the name of stable isomorphism [25], but it was not in the setting of SDSs, and was only briefly discussed. In this section, we show how cyclic shifts and reflections of the update order give rise to cycle equivalent SDS maps, and we provide a connection to the theory of Coxeter groups. The neutral networks that we construct for cycle equivalence also encode conjugacy classes of Coxeter elements, and the main theorem is an enumeration of them via a recurrence relation, strengthening a result about conjugacy of Coxeter elements from [37]. We then interpret these results in the setting of dynamics of SDSs, as well as provide a connection to other areas of mathematics where this recurrence arises, including graph polynomials, node-firing games, and quiver representations.

3.1. **Shifts and reflections of update orders.**

Definition 3.1 (Cycle equivalence)**.** Two dynamical systems $\phi\colon K_1^n \to K_1^n$ and $\psi\colon K_2^m \to K_2^m$ are *cycle equivalent* if there exists a bijection between their respective sets of periodic points $h\colon \mathsf{Per}(\phi) \to \mathsf{Per}(\psi)$ such that

$$\psi|_{\mathsf{Per}(\psi)} \circ h = h \circ \phi|_{\mathsf{Per}(\phi)} \ . \tag{3.1}$$

By restriction it is clear that cycle equivalence is a strictly weaker condition than both functional and dynamical equivalence. Let $\sigma, \rho \in S_m$ be defined by

$$\sigma = (m, m-1, \ldots, 2, 1) \ , \qquad \rho = (1, m)(2, m-1) \cdots (\lceil \tfrac{m}{2} \rceil, \lfloor \tfrac{m}{2} \rfloor + 1) \ ,$$

and let C_m and D_m be the cyclic and dihedral groups

$$C_m = \langle \sigma \rangle \quad \text{and} \quad D_m = \langle \sigma, \rho \rangle \ . \tag{3.2}$$

Both C_m and D_m act on the set of length-m words W_m by $\omega = (\omega_1, \ldots, \omega_m)$ via the action in (1.3). Define $\boldsymbol{\sigma}_s(\omega) = \sigma^s \cdot \omega$ and $\boldsymbol{\rho}(\omega) = \rho \cdot \omega = (\omega_m, \omega_{m-1}, \ldots, \omega_2, \omega_1)$, so in particular, $\boldsymbol{\sigma}_1(\omega) = \sigma \cdot \omega = (\omega_2, \ldots, \omega_m, \omega_1)$, a cyclic shift of ω. The motivation for defining these actions is apparent by the following theorem.

Theorem 3.2. *For any $\omega \in W_Y$, the* SDS *maps $[\mathfrak{F}_Y, \omega]$ and $[\mathfrak{F}_Y, \boldsymbol{\sigma}_s(\omega)]$ are cycle equivalent.*

Proof. Set $P_k = \mathsf{Per}[\mathfrak{F}_Y, \boldsymbol{\sigma}_k(\omega)]$. By the definition of an SDS map, the following diagram commutes

$$\begin{tikzcd}
P_{k-1} \arrow[r, "{[\mathfrak{F}_Y, \boldsymbol{\sigma}_{k-1}(\omega)]}"] \arrow[d, "F_{\omega(k)}"'] & P_{k-1} \arrow[d, "F_{\omega(k)}"] \\
P_k \arrow[r, "{[\mathfrak{F}_Y, \boldsymbol{\sigma}_k(\omega)]}"'] & P_k
\end{tikzcd} \tag{3.3}$$

for all $1 \leq k \leq m = |\omega|$. Thus we obtain the inclusion $F_{\omega(k)}(P_{k-1}) \subset P_k$, and since the restriction map $F_{\omega(k)} \colon P_{k-1} \longrightarrow P_k$ is an injection, it follows that $|P_{k-1}| \leq |P_k|$. We therefore obtain the sequence of inequalities

$$\left|\mathsf{Per}[\mathfrak{F}_Y, \omega]\right| \leq \left|\mathsf{Per}[\mathfrak{F}_Y, \boldsymbol{\sigma}_1(\omega)]\right| \leq \cdots \leq \left|\mathsf{Per}[\mathfrak{F}_Y, \boldsymbol{\sigma}_{m-1}(\omega)]\right| \leq \left|\mathsf{Per}[\mathfrak{F}_Y, \omega]\right| ,$$

from which it follows that all inequalities are, in fact, equalities. Since the graph and state space are finite all the restriction maps $F_{\omega(k)}$ in (3.3) are bijections and the theorem follows. $\square$

Theorem 3.2 shows that the action of the cyclic group C_m on an SDS update order preserves the cycle structure of the phase space. In the case of $K = \mathbb{F}_2$, we may act on the update order by D_m. This stems from the following result from [29].

Proposition 3.3. *Let $(Y, \mathfrak{F}_Y, \omega)$ be an* SDS *over $\mathbb{F}_2$ with periodic points $P \subset \mathbb{F}_2^n$. Then*

$$\left([\mathfrak{F}_Y, \omega]\big|_P\right)^{-1} = [\mathfrak{F}_Y, \boldsymbol{\rho}(\omega)]\big|_P \, . \tag{3.4}$$

This follows from the fact that for each vertex, the vertex function f_i when restricted to the i^{th} coordinate of the set of periodic points, is a bijection for each choice of states of vertices in $N_{1,Y}(i)$. There are only two such restricted maps: the identity map $y_i \mapsto y_i$ and the map $y_i \mapsto 1 + y_i$. Thus composing the two maps in (3.4) in either order gives the identity map. The corollary below is now clear:

Corollary 3.4. *Over $K = \mathbb{F}_2$ the* SDS *maps $[\mathfrak{F}_Y, \omega]$ and $[\mathfrak{F}_Y, \boldsymbol{\rho}(\omega)]$ are cycle equivalent.*

We know that for any $g \in G = C_m$ the SDS maps $[\mathfrak{F}_Y, \omega]$ and $[\mathfrak{F}_Y, g \cdot \omega]$ are cycle equivalent, where $|\omega| = m$, and by Corollary 3.4, the same statement holds for $G = D_m$ if $K = \mathbb{F}_2$. We now have the following situation: elements π and π' with $[\pi]_Y \neq [\pi']_Y$ generally give rise to functionally non-equivalent SDS maps. However, if there exists $g \in G$, $\bar{\pi} \in [\pi]_Y$ and $\bar{\pi}' \in [\pi']_Y$ such that $g \cdot \bar{\pi} = \bar{\pi}'$, then the classes $[\pi]_Y$ and $[\pi']_Y$ give rise to cycle equivalent SDS maps.

3.2. **Neutral networks for cycle equivalence.** In this section, we define two graphs over $S_Y/\sim_Y$ whose connected components give rise to cycle equivalent SDS maps for a fixed sequence of functions $\mathfrak{F}_Y$. For ease of notation we will consider permutation SDSs, but it is not difficult to see how to extend this to systems with general word update orders. Since cycle equivalence is a coarsening of functional equivalence, it is natural to construct these graphs using $S_Y/\sim_Y$ rather than S_Y as the vertex set. Let

$C(Y)$ and $D(Y)$ be the undirected graphs defined by

$$\mathrm{v}[C(Y)] = S_Y/{\sim_Y}\ , \qquad \mathrm{e}[C(Y)] = \big\{\{[\pi]_Y, [\boldsymbol{\sigma}_1(\pi)]_Y\} \mid \pi \in S_Y\big\}\ ,$$

$$\mathrm{v}[D(Y)] = S_Y/{\sim_Y}\ , \qquad \mathrm{e}[D(Y)] = \big\{\{[\pi]_Y,\, [\boldsymbol{\rho}(\pi)]_Y\} \ \mid \pi \in S_Y\big\} \cup \mathrm{e}[C(Y)]\ .$$

Define $\kappa(Y)$ and $\delta(Y)$ to be the number of connected components of $C(Y)$ and $D(Y)$, respectively. By construction, $C(Y)$ is a subgraph of $D(Y)$ and $\delta(Y) \le \kappa(Y)$. From Theorem 3.2 it is clear that $\kappa(Y)$ is an upper bound for the number of different SDS cycle equivalence classes obtainable through all possible choice of simple update order. For $K = \mathbb{F}_2$ it follows from Proposition 3.3 that $\delta(Y)$ is an upper bound as well. It is straightforward to extend the definitions of $C(Y)$ and $D(Y)$ to the infinite graphs $\hat{C}(Y)$ and $\hat{D}(Y)$ for the case of general word update orders from W_Y, but we will stick with the case of simple update orders here.

Example 3.5. The update graph $U(\mathsf{Circ}_4)$ is shown in Figure 2.1. The graphs $C(\mathsf{Circ}_4)$ and $D(\mathsf{Circ}_4)$ are displayed here in Figure 3.1 where the dashed lines are edges that belong to $D(\mathsf{Circ}_4)$ but not to $C(\mathsf{Circ}_4)$. The

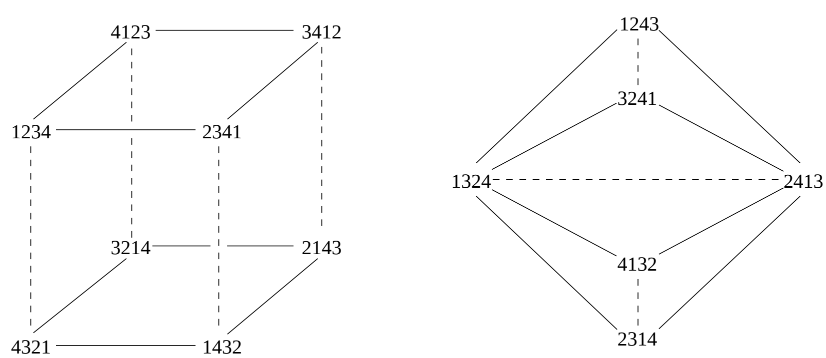


FIGURE 3.1. The graphs $C(\mathsf{Circ}_4)$ and $D(\mathsf{Circ}_4)$. The dashed lines are edges in $D(\mathsf{Circ}_4)$ but not in $C(\mathsf{Circ}_4)$.

vertices in Figure 3.1 are labeled by a permutation in the corresponding equivalence class in $S_Y/{\sim_Y}$. The vertices of the cube-shaped component are all singletons in $S_Y/{\sim_Y}$ (see Figure 2.1). The equivalence classes $[1324]_{\mathsf{Circ}_4}$ and $[2413]_{\mathsf{Circ}_4}$ both consist of four permutations, while the remaining four vertices on that component are equivalence classes with two permutations each. Clearly, $\kappa(\mathsf{Circ}_4) = 3$ and $\delta(\mathsf{Circ}_4) = 2$.

Example 3.6. Let Y be the complete bipartite graph $K_{2,3}$, where the partition of the vertex set is $\{\{1,3,5\},\{2,4\}\}$. The graph $U(K_{2,3})$ is shown in Figure 3.2 with vertex labels omitted. By simply counting the components we see that $\alpha(K_{2,3}) = 46$. We can better understand the component struc-

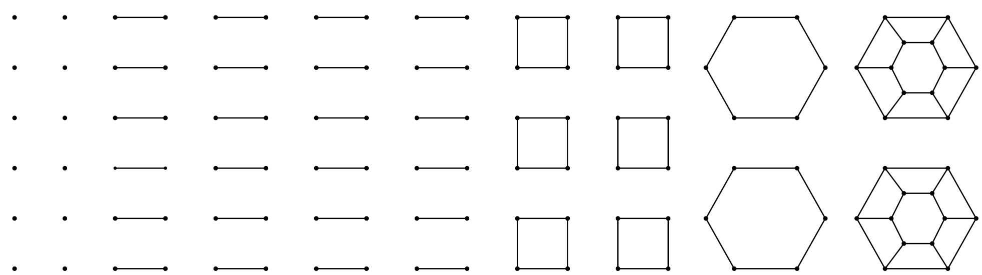


FIGURE 3.2. The update graph $U(K_{2,3})$.

ture of $U(K_{2,3})$ by mapping permutations as $(\pi_i)_i \overset{\phi}{\mapsto} (\pi_i \mod 2)_i$. Non-adjacency in Y coincides with parity, that is, if $\pi \sim_Y \sigma$, then $\phi(\pi) = \phi(\sigma)$. Through the map ϕ we see that the 12 singleton points in $U(K_{2,3})$ are precisely those with image 10101. Each of the 24 size-two components correspond to a pair of permutations with ϕ-image of the form 01011, 11010, 01101, or 10110. The six square-components arise from the permutations with ϕ-image 10011 and 11001. Finally, the permutations in the two hexagon-components are of the form 01110, and those in the two largest components have ϕ-image of the form 11100 or 00111.

The graphs $C(K_{2,3})$ and $D(K_{2,3})$ are shown in Figure 3.3. The dashed lines are edges that belong to $D(K_{2,3})$ but not to $C(K_{2,3})$. The vertices in

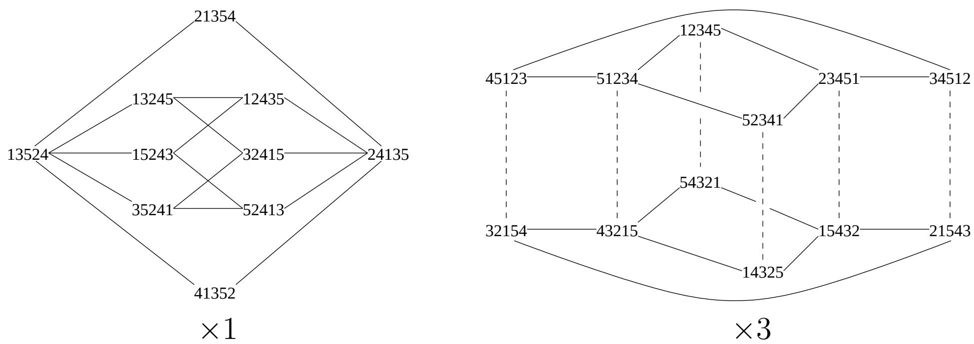


FIGURE 3.3. The graph $C(K_{2,3})$ contains the component on the left, and three isomorphic copies of the structure on the right (but with different vertex labels). The dashed lines are edges in $D(K_{2,3})$ but not in $C(K_{2,3})$.

Figure 3.3 are labeled by a permutation in the corresponding equivalence class in $S_Y/\sim_Y$. There are three isomorphic copies of the component on the right, but only one is shown. Each of these three components contains permutations whose ϕ-image is in $\{01101, 11010, 10101, 01011, 10110\}$. The component on the left contains all of the remaining permutations, i.e., all π for which $\phi(\pi) \in \{11100, 11001, 10011, 00111, 01110\}$. Clearly, $\kappa(K_{2,3}) = 7$ and $\delta(K_{2,3}) = 4$.

3.3. **Source-to-sink operations.** In this section, we show how the component structure of $C(Y)$ is precisely captured by a certain source-to-sink operation on the acyclic orientations of Y. This also arises in the setting of conjugacy classes of Coxeter elements, and we will briefly introduce the basic concepts to demonstrate this connection.

The bijection in (2.1) identifies $[\pi]_Y$ with the acyclic orientation O_Y^{π}. Take $\pi \in [\pi']_Y$. It is clear that mapping π to $\boldsymbol{\sigma}_1(\pi)$ corresponds precisely to converting the vertex π_1 from a source to a sink in O_Y^{π}, which we call a *source-to-sink operation*, as in [37], or a *click*. Two orientations $O_Y, O'_Y \in \mathsf{Acyc}(Y)$ where O_Y can be transformed into O'_Y by a sequence of clicks are said to click-related, and we write this as $\mathbf{c}(O_Y) = O'_Y$ where $\mathbf{c} = c_1c_2\cdots c_k$ and $c_i \in \mathrm{v}[Y]$. By this observation and with Theorem 3.2, update orders belonging to equivalence classes whose corresponding acyclic orientations are click-related give rise to cycle equivalent SDS maps. It is elementary to verify that the click relation is an equivalence relation on $\mathsf{Acyc}(Y)$, and we denote it by $\sim_\kappa$.

The edges in $C(Y)$ correspond with single source-to-sink operations, and thus the number of equivalence classes of $\mathsf{Acyc}(Y)$ under the source-to-sink relation is simply $\kappa(Y)$, the number of connected components of $C(Y)$. Update orders from $\sim_Y$ classes belonging to the same component in $C(Y)$ are said to be *κ-equivalent*, as are the corresponding acyclic orientations. For two κ-equivalent orders π and π' there is a sequence of adjacent non-edge transpositions and cyclic shifts that map π to π'. This is simply a consequence of the definition of $S_Y/\!\sim_Y$ and $C(Y)$. From here there is a close connection to the enumeration of conjugacy classes of Coxeter elements, as will be explained in the next section.

3.4. **Coxeter groups and Coxeter elements.** A *Coxeter group* is a group with presentation

$$\langle s_1, \ldots, s_n \mid (s_i s_j)^{m_{ij}} \rangle$$

where $m_{ij} = 1$ if $i = j$ and $m_{ij} \geq 2$ otherwise. The generators are involutions, and so a Coxeter group is in a sense a generalized reflection group. Given a Coxeter group, the matrix $M = (m_{ij})$ is the *Coxeter matrix*, and the graph with vertex set $\{s_1, \ldots, s_n\}$ and edge set $\{\{s_i, s_j\} \mid m_{ij} \geq 3\}$ with edges labels m_{ij} is the *Coxeter graph*. Disregarding the edge labels we see that there is a close connection between generators of a Coxeter group, and the Coxeter graph on the one hand, and the Y-local functions, and SDS base graph Y on the other hand. For example, generators s_i and s_j for which $\{s_i, s_j\}$ is not an edge commute, and in the same way, Y-local maps F_i and F_j commute if $\{i, j\}$ is not an edge in Y.

A *Coxeter element* [36] is a product of the generators in some order, i.e.,

$$\prod_{i=1}^{n} s_{\pi(i)} = s_{\pi(1)} s_{\pi(2)} \cdot \cdots \cdot s_{\pi(n)} \ , \tag{3.5}$$

and thus there is a correspondence between the set of Coxeter elements and the set of permutation SDS maps over the Coxeter graph for a fixed sequence of vertex functions. Explicitly, for a fixed sequence of functions $\mathfrak{F}_Y$, there is a surjection from the set of Coxeter elements of the group with Coxeter graph Y, to the set of permutation SDS maps, defined by

$$s_{\pi(1)} s_{\pi(2)} \cdot \cdots \cdot s_{\pi(n)} \longmapsto [\mathfrak{F}_Y, \pi] \ .$$

In general, this map need not be injective, and taking $\mathfrak{F}_Y$ to be the sequence of identity functions is a simple example of this. However, Proposition 2.2 implies that that it is a bijection for the Nor_Y local functions. In light of the correspondence between Coxeter elements, components of $U(Y)$, and permutation SDS maps, it is not surprising that there is also a bijection between the set of Coxeter elements of a Coxeter group and the set of acyclic orientations of the Coxeter graph [20]. If we conjugate a Coxeter element $\prod s_{\pi(i)}$ by $s_{\pi(1)}$ we get

$$s_{\pi(1)}\big(s_{\pi(1)}s_{\pi(2)}\cdots s_{\pi(n)}\big)s_{\pi(1)} = s_{\pi(2)}\cdots s_{\pi(n)}s_{\pi(1)}\ , \tag{3.6}$$

and thus $\prod s_{\pi(i)}$ and $\prod s_{\boldsymbol{\sigma}_k(\pi(i))}$ are conjugate for all $k \in \mathbb{Z}$. It is not known whether conjugation by "source generators" (or cyclically shifting) is generally sufficient to fully characterize conjugacy classes of Coxeter elements. However, it does hold for special cases [37], and the general case is believed to be true as well [14].

3.5. **The case $K = \mathbb{F}_2$.** For SDSs over $K = \mathbb{F}_2$, there is an additional operation on acyclic orientations that leads to cycle equivalent SDS maps. By Proposition 3.3 we may in this case also consider the group D_n and the associated graph $D(Y)$. Through the bijection in (2.1), this additionally identifies O_Y^{π} with the reverse orientation $O_Y^{\boldsymbol{\rho}(\pi)}$, the unique orientation that satisfies $O_Y^{\pi}(\{i,j\}) \neq O_Y^{\boldsymbol{\rho}(\pi)}(\{i,j\})$ for every $\{i,j\} \in \mathrm{e}[Y]$. Two permutations belonging to $\sim_Y$ classes on the same connected component in $D(Y)$ are called *δ-equivalent*, as are their corresponding acyclic orientations. For δ-equivalence of acyclic orientations we extend the notion of a click sequence to include an element encoding reversal of all orientations. Such a sequence is called an *extended click sequence*. This relation, being a consequence of $K = \mathbb{F}_2$, does not seem to play any role in Coxeter theory. It would correspond to identifying two conjugacy classes if for some Coxeter element g, one contains g and the other contains g^{-1}, and this is not a natural group theoretic construction.

The following result gives insight into the component structure of the graph $C(Y)$.

Proposition 3.7. *Let Y be a connected graph on n vertices and let $g, h \in C_n$ with $g \neq h$. Then $[g \cdot \pi]_Y \neq [h \cdot \pi]_Y$.*

Proof. Assume $g \neq n$ with $[g \cdot \pi]_Y = [h \cdot \pi]_Y$. By construction, we have $g \cdot \pi = \boldsymbol{\sigma}_s(\pi)$ and $h \cdot \pi = \boldsymbol{\sigma}_{s'}(\pi)$. Without loss of generality we may assume $s' < s$. Let $V' \subset V = \mathrm{v}[Y]$ be the initial subsequence of vertices in $\boldsymbol{\sigma}_{s'}(\pi)$ that occurs at the end in $\boldsymbol{\sigma}_s(\pi)$. If any of the vertices in V' were adjacent to any of the vertices in $V \setminus V'$ in Y it would imply that $[\boldsymbol{\sigma}_s(\pi)]_Y \neq [\boldsymbol{\sigma}_{s'}(\pi)]_Y$. The only possibility is that Y is not connected, but this contradicts the assumptions of the proposition. $\square$

The result in Proposition 3.7 additionally holds for D_n as long as Y is not bipartite.

Proposition 3.8. *Let Y be a connected graph on n vertices and let $g, h \in D_n$ with $g \neq h$. If $[g \cdot \pi]_Y = [h \cdot \pi]_Y$ holds then Y must be bipartite.*

Proof. From $[g \cdot \pi]_Y = [h \cdot \pi]_Y$ it follows from Proposition 3.7 that g and h lie in different cosets of C_n in D_n. Without loss of generality we may assume that $g = \sigma^s$ and $h = \rho\sigma^{s'}$. Let $m = |s' - s|$ and $m' = n - m$. If $s' > s$ (resp. $s' < s$) the first (resp. last) m elements of $g \cdot \pi$ and $h \cdot \pi$ are the same but occur in reverse order. Call the set of these elements V_1. The remaining m' elements occur in reverse order as well in the two permutations. Let V_2 denote the set of these elements. For $[g \cdot \pi]_Y = [h \cdot \pi]_Y$ to hold, there cannot be an edge between any two vertices in V_1, or between any two vertices in V_2. Therefore, the graph Y must be a subgraph of $K(V_1, V_2)$, the complete bipartite graph with vertex sets V_1 and V_2. □ □

Remark 3.9. The pairs $(\sigma^s, \rho\sigma^{s'})$ and $(\sigma^{s'}, \rho\sigma^s)$ determine the same bipartite graph in the proof above, with the only difference being that V_1 and V_2 switch roles. Moreover, if Y is bipartite, then the equality $[\pi] = [\boldsymbol{\rho}(\pi)] \in S_Y/{\sim_\delta}$ can only hold for one such class $[\pi]_Y \in S_Y/{\sim_\kappa}$, namely the class that contains the ordering π where the vertices V_1 appear consecutively, as the initial or terminal subword of π.

From Propositions 3.7 and 3.8, we can derive upper bounds on $\kappa(Y)$ and $\delta(Y)$, and thus on the number of SDS maps up to cycle equivalence.

Corollary 3.10. *If Y is a connected undirected graph on n vertices, then $\kappa(Y) \leq \frac{1}{n}\,\alpha(Y)$. Moreover, if Y is not bipartite, then $\delta(Y) \leq \frac{1}{2n}\,\alpha(Y)$.*

Proof. By Proposition 3.7, for any $\pi \in S_Y$, the set $\{O_Y^{\boldsymbol{\sigma}_1(\pi)}, \dots, O_Y^{\boldsymbol{\sigma}_n(\pi)}\}$ contains n distinct acyclic orientations that are all κ-equivalent. The first statement now follows. By Proposition 3.8, if Y is not bipartite, then for any $\pi \in S_Y$, the set $\{O_Y^{\boldsymbol{\sigma}_1(\pi)}, \dots, O_Y^{\boldsymbol{\sigma}_n(\pi)}\} \cup \{O_Y^{\boldsymbol{\rho\sigma}_1(\pi)}, \dots, O_Y^{\boldsymbol{\rho\sigma}_n(\pi)}\}$ contains

$2n$ distinct acyclic orientations that are all δ-equivalent. This establishes the second statement. $\square$

3.6. **Analysis of $\kappa(Y)$ and $\delta(Y)$.** In this section we analyze the functions $\kappa(Y)$ and $\delta(Y)$ that count the number of components of $C(Y)$ and $D(Y)$, respectively. We first show that $\delta(Y)$ is determined by $\kappa(Y)$.

Lemma 3.11. *The map $\boldsymbol{\rho}\colon S_Y \longrightarrow S_Y$ extends to an involution*

$$\boldsymbol{\rho}^*\colon \mathsf{Acyc}(Y)/\sim_\kappa \longrightarrow \mathsf{Acyc}(Y)/\sim_\kappa \ . \tag{3.7}$$

Proof. Let $\pi \in S_Y$, and let O_Y^π be the corresponding acyclic orientation. It is elementary to verify that the diagram

$$\begin{tikzcd} \pi \arrow[r, "\iota"] \arrow[d, "\boldsymbol{\rho}"'] & O_Y^\pi \arrow[r, "\iota_{\sim_\kappa}"] \arrow[d, "\hat{\boldsymbol{\rho}}"] & {[O_Y^\pi]} \arrow[d, "\boldsymbol{\rho}^*"] \\ \boldsymbol{\rho}(\pi) \arrow[r, "\iota"] & O_Y^{\boldsymbol{\rho}(\pi)} \arrow[r, "\iota_{\sim_\kappa}"] & {[O_Y^{\boldsymbol{\rho}(\pi)}]} \end{tikzcd} \tag{3.8}$$

commutes, and the lemma follows. $\square$

Because $\boldsymbol{\rho}^*$ is an involution, it follows immediately that $\delta(Y) = \frac{1}{2}\kappa(Y)$ if and only if $\boldsymbol{\rho}^*$ has no fixed points. The next result shows that this happens precisely when Y contains an odd cycle.

Proposition 3.12. *Let Y be a connected undirected graph. If Y is not bipartite then $\delta(Y) = \frac{1}{2}\kappa(Y)$. If Y is bipartite then $\delta(Y) = \frac{1}{2}(\kappa(Y)+1)$.*

Proof. By Proposition 3.8, if Y is not bipartite, then $\boldsymbol{\rho}^*$ has no fixed points. This, with the fact that $\boldsymbol{\rho}^*$ is an involution, establishes the first statement.

The second statement holds because by Remark 3.9, if Y is connected and bipartite, $\boldsymbol{\rho}^*$ has precisely one fixed point. $\square$

Worded differently, Proposition 3.12 says that $\delta(Y) = \lceil \kappa(Y)/2 \rceil$. Moreover, it implies the following corollary.

Corollary 3.13. *A connected graph Y is bipartite if and only if $\kappa(Y)$ is odd.*

For examples where $\boldsymbol{\rho}^*$ has a fixed point see Figure 3.1 in Example 3.5 and Figure 3.3 in Example 3.6. Now that we know how to compute $\delta(Y)$ given Y and $\kappa(Y)$, we shift our attention to computing $\kappa(Y)$. The quantity $\kappa(Y)$ is multiplicative in the following sense:

Proposition 3.14. *Let Y be the disjoint union of undirected graphs Y_1 and Y_2. Then*

$$\kappa(Y) = \kappa(Y_1)\kappa(Y_2) .$$

Proof. Every element of $\mathsf{Acyc}(Y)/\sim_\kappa$ can be represented by a unique pair of elements in $\mathsf{Acyc}(Y_1)/\sim_\kappa$ and $\mathsf{Acyc}(Y_2)/\sim_\kappa$, and conversely. $\square$

In light of this result we may assume that Y is connected when computing $\kappa(Y)$. It is easy to extend Proposition 3.14 to the case when Y_1 and Y_2 are connected by a single edge.

Definition 3.15 (Bridges and cycle-edges)**.** An edge e of a graph Y is a *bridge* if removing e increases the number of components of Y. Otherwise, it is a *cycle-edge.* We note that bridges are precisely the edges that are not contained in a cycle. The subgraph of Y obtained by removing all bridges is denoted by $\mathrm{Cycle}(Y)$.

The next result shows that bridges do not contribute to $\kappa(Y)$.

Proposition 3.16. *Let Y be a connected undirected graph, and $e \in \mathrm{e}[Y]$ a bridge. Denote the two components of the graph with e removed by Y_1 and Y_2. Then*

$$\kappa(Y) = \kappa(Y_1)\kappa(Y_2)\ .$$

Proof. There is a 2–1 correspondence of acyclic orientations of Y, and acyclic orientations of $Y_1 \cup Y_2$, by the restriction map. We will show that if O_Y and O'_Y differ only on the edge $e = \{v_1, v_2\}$ then $O_Y \sim_\kappa O'_Y$. The result will then follow from Proposition 3.14. Without loss of generality, assume that e is oriented (v_1, v_2). Any click sequence containing every vertex in Y_1 precisely once, and no vertices of Y_2, carries O_Y to O'_Y, and the result follows. □

Proposition 3.14 together with Proposition 3.16 implies that $\kappa(Y)$ is preserved under removal of a bridge edge. By iterating this we get the following result.

Corollary 3.17. *If $Z = \mathrm{Cycle}(Y)$, then $\kappa(Y) = \kappa(Z)$ and $\delta(Y) = \delta(Z)$.*

Proof. The first equality follows from a simple induction argument, with Proposition 3.16. The second equality holds because

$$\delta(Y) = \lceil \kappa(Y)/2 \rceil = \lceil \kappa(Z)/2 \rceil = \delta(Z)\ .$$

□

We remark that a special case of the first equality in Corollary 3.17, when $\mathrm{Cycle}(Y) = \mathsf{Circ}_n$, appeared in [37]. We also immediately get the following:

Corollary 3.18. *If Y is forest, then $\kappa(Y) = \delta(Y) = 1$.*

Proof. If Y is forest, then $\mathrm{Cycle}(Y) = E_n$, the empty graph on n vertices. Since $\kappa(E_n) = 1$ the result now follows from Corollary 3.17. $\square$

A special case of this result in Coxeter theory says that all Coxeter elements of a finite Coxeter group are conjugate [20]. This holds because the Coxeter graph of a finite Coxeter group is always a tree. In the setting of SDSs, κ- and δ-equivalent update orders give rise to cycle equivalent maps. Thus we can make the following statement about SDS maps where the base graph is a forest.

Proposition 3.19. *Let Y be a forest and $\mathfrak{F}_Y$ be a sequence of Y-local functions, and $\omega \in W_Y$. For any word ω with $|\omega| = m$, and $g \in S_m$, the* SDS *maps $[\mathfrak{F}_Y, \omega]$ and $[\mathfrak{F}_Y, g \cdot \omega]$ are cycle equivalent.*

In other words, when Y is a forest, the cycle structure is unchanged under permutations of the update order. This result may not be that significant if there are only a few periodic points. However, for other functions, such as invertible ones, it is very powerful.

Corollary 3.20. *Let Y be a forest, and $\mathfrak{F}_Y = (F_i)_{i \in \mathrm{v}[Y]}$ any sequence of Y-local functions such that each F_i is invertible. Then for any $\omega \in W_Y$ with $|\omega| = m$, and $g \in S_m$, the phase spaces $\Gamma[\mathfrak{F}_Y, \omega]$ and $\Gamma[\mathfrak{F}_Y, g \cdot \omega]$ are isomorphic.*

The parity functions $\mathsf{par}_k \colon \mathbb{F}_2^k \longrightarrow \mathbb{F}_2$ are defined as $\mathsf{par}_k(\mathbf{y}) = \sum_i y_i$, modulo 2, and their negations, $\overline{\mathsf{par}}_k \colon \mathbb{F}_2^k \longrightarrow \mathbb{F}_2$, defined be $\overline{\mathsf{par}}_k(\mathbf{y}) = \sum_i 1+$

y_i, modulo 2, are examples of vertex functions whose corresponding Y-local functions are invertible for any graph Y. It should be noted that the converse of Corollary 3.18 holds as well. This is due to the fact that $\kappa(Y)$ satisfies the same recurrence as $\alpha(Y)$ in (2.2), with the added condition that e must be a cycle-edge. We will formally state and prove this in the following section, but first, we need to develop a better understanding of the structure of the κ-equivalence classes of a graph.

3.7. **Poset structure of κ-equivalence classes.** In this section, we will show how one may associate a poset to the κ-equivalence classes of a graph. The properties of this poset give us better insight into the structure of $\mathsf{Acyc}(Y)/\sim_\kappa$. Additionally, it allows us to construct a recurrence for $\kappa(Y)$ under deletion and contraction of cycle-edges.

Throughout, we will let $e = \{v, w\}$ be a fixed cycle-edge of the connected graph Y, and for ease of notation we set $Y' = Y'_e$ and $Y'' = Y''_e$. For $O_Y \in \mathsf{Acyc}(Y)$ we let $O_{Y'}$ and $O_{Y''}$ denote the induced orientations of Y' and Y''. Notice that $O_{Y'}$ is always acyclic, while $O_{Y''}$ is acyclic if and only if there is no directed path with endpoints v and w in $O_{Y'}$. Finally, we let $[O_Y]$ denote the κ-equivalence class containing O_Y.

The interval $[a, b]$ of a poset $\mathcal{P}$ (where $a \leq b$) is the subposet consisting of all $c \in \mathcal{P}$ such that $a \leq c \leq b$. Viewing a finite poset $\mathcal{P}$ as a directed graph $D_{\mathcal{P}}$, the interval $[a, b]$ contains precisely the vertices that lie on a directed path from a to b, and thus is a vertex-induced subgraph of $D_{\mathcal{P}}$. By assumption, Y contains the edge $\{v, w\}$, so for all $O_Y \in \mathsf{Acyc}(Y)$ either

$v \leq_{O_Y} w$ or $w \leq_{O_Y} v$. In this section, we will study the interval $[v, w]$ in the poset O_Y (when $v \leq_{O_Y} w$) and its behavior under clicks.

Definition 3.21 (vw-interval)**.** Let $\mathsf{Acyc}_{\leq}(Y)$ be the set of acyclic orientations of vertex-induced subgraphs of Y. We define the map

$$\mathcal{I} \colon \mathsf{Acyc}(Y) \longrightarrow \mathsf{Acyc}_{\leq}(Y) ,$$

by $\mathcal{I}(O_Y) = [v, w]$ if $v \leq_{O_Y} w$, and by $\mathcal{I}(O_Y) = \emptyset$ (the null graph) otherwise. When $\mathcal{I}(O_Y) \neq \emptyset$ we refer to $\mathcal{I}(O_Y)$ as the *vw-interval* of O_Y.

Elements of $\mathsf{Acyc}_{\leq}(Y)$ can be thought of as subposets of $\mathsf{Acyc}(Y)$. Through a slight abuse of notation, we will at times refer to $\mathcal{I}(O_Y)$ as a poset, a directed graph, or a subset of $\mathrm{v}[O_Y]$. In this last case, it is understood that the relations are inherited from O_Y.

For an undirected path $P = v_1, v_2, \ldots, v_k$ in Y, we define the function

$$\nu_P \colon \mathsf{Acyc}(Y) \longrightarrow \mathbb{Z} ,$$

where $\nu_P(O_Y)$ is the number of edges oriented as (v_i, v_{i+1}) in O_Y, minus the number of edges oriented as (v_{i+1}, v_i). It is clear that if P is a cycle, then ν_P is preserved under clicks, and in this case, ν_P extends to a map $\nu_P^* \colon \mathsf{Acyc}(Y)/{\sim_\kappa} \longrightarrow \mathbb{Z}$.

We will now prove a series of structural results about the vw-interval. Since $\{v, w\} \in \mathrm{e}[Y]$, every κ-class contains at least one orientation O_Y with $v \leq_{O_Y} w$, and thus there is at least one element O_Y in each κ-class with $\mathcal{I}(O_Y) \neq \emptyset$. The next results shows how we can extend the notion of vw-interval from over $\mathsf{Acyc}(Y)$ to $\mathsf{Acyc}(Y)/{\sim_\kappa}$.

Proposition 3.22. *The map $\mathcal{I}$ can be extended to a map*

$$\mathcal{I}^*\colon \mathsf{Acyc}(Y)/{\sim_\kappa} \longrightarrow \mathsf{Acyc}_{\leq}(Y) \quad by \quad \mathcal{I}^*([O_Y]) = \mathcal{I}(O_Y^1)\ ,$$

where O_Y^1 is any element of $[O_Y]$ for which $\mathcal{I}(O_Y^1) \neq \emptyset$.

Proof. It suffices to prove that $\mathcal{I}^*$ is well-defined. Consider $O_Y^1 \sim_\kappa O_Y^2$ with $v \leq_{O_Y^i} w$ for $i = 1, 2$. To show that $\mathcal{I}(O_Y^1) = \mathcal{I}(O_Y^2)$ let a be a vertex in $\mathcal{I}(O_Y^1)$. Then a lies on a directed path P' from v to w in O_Y^1, say of length $k \geq 2$. Let P be the cycle formed by adding vertex v to the end of P'. Clearly $\nu_P(O_Y^1) = k - 1$ since $O_Y^1(e) = (v, w)$.

By assumption, $O_Y^2 \in [O_Y^1]$ with $v \leq_{O_Y^2} w$. Since ν_P is constant on $[O_Y^1]$ it follows from $\nu_P(O_Y^1) = k - 1 = \nu_P(O_Y^2)$ that every edge of P' is oriented identically in O_Y^1 and O_Y^2, and hence that every directed path P' in O_Y^1 is contained in O_Y^2 as well. The reverse inclusion follows by an identical argument. □

In light of Proposition 3.22, we define the vw-interval of a κ-class $[O_Y]$ to be $\mathcal{I}^*([O_Y])$. The vw-interval will be central in understanding properties of click-sequences. First, we will make a simple observation without proof, which also appears in [39] in the context of admissible sequences in Coxeter theory.

Proposition 3.23. *Let $O_Y \in \mathsf{Acyc}(Y)$, let $\mathbf{c} = c_1 c_2 \cdots c_m$ be an associated click-sequence, and consider any directed edge (v_1, v_2) in O_Y. Then the occurrences of v_1 and v_2 in $\mathbf{c}$ alternate, with v_1 appearing first.*

Because $\{v, w\} \in \mathrm{e}[Y]$, we can say more about the vertices in $\mathcal{I}(O_Y)$ that appear between successive instances in v and w in a click-sequence.

Proposition 3.24. *Let $O_Y \in \mathsf{Acyc}(Y)$, and let $\mathbf{c} = c_1 c_2 \cdots c_m$ be an associated click-sequence that contain every vertex of $\mathcal{I}(O_Y)$ at least once and with $c_1 = v$. Then every vertex of $\mathcal{I}(O_Y)$ appears in $\mathbf{c}$ before any vertex in $\mathcal{I}(O_Y)$ appears twice.*

Proof. The proof is by contradiction. Assume the statement is false, and let $a \in \mathcal{I}(O_Y)$ be the first vertex whose second instance in $\mathbf{c}$ occurs before the first instance of some other vertex $z \in \mathcal{I}(O_Y)$. If $a \neq v$, then a is not a source in O_Y, and there exists a directed edge (a', a). By Proposition 3.23, a' must appear in $\mathbf{c}$ before the first instance of a, but also between the two first instances of a. This is impossible, because a was chosen to be the first vertex appearing twice in $\mathbf{c}$. That only leaves $a = v$, and v must appear twice before the first instance of w. However, this contradicts the statement of Proposition 3.23 because $\{v, w\} \in \mathrm{e}[Y]$. □

The next result shows that for any click-sequence $\mathbf{c}$ that contains every element in $\mathcal{I}(O_Y)$ precisely once, we may assume without loss of generality that the vertices in $\mathcal{I}(O_Y)$ appear consecutively.

Proposition 3.25. *Let $O_Y \in \mathsf{Acyc}(Y)$ be an acyclic orientation with $v \leq_{O_Y} w$. If $\mathbf{c} = c_1 c_2 \cdots c_m$ is an associated click-sequence containing precisely one instance of w, and no subsequent instances of vertices from $\mathcal{I}(O_Y)$, then there exists a click-sequence $\mathbf{c}' = c'_1 c'_2 \cdots c'_m$ such that (i) there exists an interval $[p, q]$ of $\mathbb{N}$ with $c'_j \in \mathcal{I}(O_Y)$ iff $p \leq j \leq q$, and (ii) $\mathbf{c}(O_Y) = \mathbf{c}'(O_Y)$.*

Proof. We prove the proposition by constructing a desired click-sequence $\mathbf{c}''$ from $\mathbf{c}$ through a series of transpositions where each intermediate click-sequence $\mathbf{c}'$ satisfies $\mathbf{c}(O_Y) = \mathbf{c}'(O_Y)$. Such transpositions are said to have property T.

Let $I = \mathcal{I}(O_Y)$, and let A be the set of vertices in $I^c = \mathrm{v}[Y] \setminus I$ that lie on a directed path in O_Y to a vertex in I (vertices *above* I), and let B be the set of vertices that lie on a directed path in O_Y from a vertex in I (vertices *below* I). Let C be the complement of $I \cup A \cup B$. Two vertices $c_i, c_j \in A \cup B$ with $i < j$ for which there is no element $c_k \in A \cup B$ with $i < k < j$ are said to be *tight*. We will investigate when transpositions of tight vertices in a click-sequence $\mathbf{c}$ of O_Y has property T, and we will see that this is always the case if $c_i \in B$ and $c_j \in A$. Consider the intermediate acyclic orientation after applying successive clicks $c_1 c_2 \cdots c_{i-1}$ to O_Y. Obviously, c_i is a source. At this point, if c_j were not a source, then there would be an adjacent vertex $a \in A$ with the edge $\{a, c_j\}$ oriented (a, c_j). For c_j to be clicked as usual (i.e., as a source), a must be clicked first, but this would break the assumption that c_i and c_j are tight. Therefore, c_i and c_j are both sources at this intermediate step, and so the vertices $c_i, c_{i+1}, \ldots, c_j$ are an independent set of sources, and may be permuted in any manner without changing the image of the click sequence. Therefore, the transposition of c_i and c_j in $\mathbf{c}$ has property T, as claimed. By iteratively transposing tight pairs in $\mathbf{c}$, we can construct a click-sequence with the property that every vertex in A comes before every vertex in B. In light of this, we may assume without loss of generality that $\mathbf{c}$ has this property.

The next step is to show that we can move all vertices in A before v, and all vertices in B after w via transpositions having property T. Let a be the first vertex in A appearing after v in the click sequence $\mathbf{c}$. We claim that the transposition moving a to the position directly preceding v has property T. This is immediate from the observation that when v is to be clicked, a is a source as well, by the definition of A, thus it may be clicked before v, without preventing subsequent clicks of vertices up until the original position of a. Therefore, we may one-by-one move the vertices in A that are between v and w, in front of v. An analogous argument shows that we may move the vertices in B that appear before w to a position directly following w. In the resulting click-sequence $\mathbf{c}'$, the only vertices between v and w are either in I or C. The subgraph of the directed graph O_Y induced by C is a disjoint union of weakly connected components, and none of the vertices are adjacent to I. By definition of A and B, there cannot exist a directed edge (c, a) or (b, c), where $a \in A$, $b \in B$, and $c \in C$. Thus for each weakly connected component of C, the vertices in the component can be moved within $\mathbf{c}'$, preserving their relative order, to a position either (i) directly after the vertices in A and before v, or (ii) directly after w and before the vertices of B. Call this resulting click-sequence $\mathbf{c}''$. As we just argued, all the transpositions occurring in the rearrangement $\mathbf{c} \mapsto \mathbf{c}''$ has property T, and $\mathbf{c}''$ contains all of the vertices in I in consecutive order, and this proves the result. $\square$

We remark that the last two results together imply that for the interval $[p,q]$ in the statement of Proposition 3.25, $c_p = v$, $c_q = w$, and the sequence $c_p c_{p+1} \cdots c_q$ contains every vertex in $\mathcal{I}(O_Y)$ precisely once. A simple induction argument implies the following.

Corollary 3.26. *Suppose that $O_Y \in \mathsf{Acyc}(Y)$ with $v \leq_{O_Y} w$, and let $\mathbf{c} = c_1 c_2 \cdots c_m$ be a click-sequence where w appears k times. Then there exists a click-sequence $\mathbf{c}' = c'_1 c'_2 \cdots c'_m$ such that (i) there are k disjoint intervals $[p_i, q_i]$ of $\mathbb{N}$ such that $c_j \in \mathcal{I}(O_Y)$ iff $p_i \leq j \leq q_i$ for some i, and (ii) $\mathbf{c}(O_Y) = \mathbf{c}'(O_Y)$.*

Proof. The argument is by induction on k. When $k = 1$, the statement is simply Proposition 3.25. Suppose the statement holds for all $k \leq N$, for some $N \in \mathbb{N}$, and let $\mathbf{c}$ be a click-sequence containing $N+1$ instances of w. Let c_ℓ be the second instance of v in $\mathbf{c}$, and consider the two click-sequences $\mathbf{c}_i := c_1 c_2 \cdots c_{\ell-1}$ and $\mathbf{c}_f := c_\ell c_{\ell+1} \cdots c_m$. By Proposition 3.25, there exists an interval $[p_1, q_1]$ with $p_1 < q_1 < \ell$, and by the induction hypothesis, there exists k intervals $[p_2, q_2], \ldots, [p_{k+1}, q_{k+1}]$ with $\ell \leq p_2 < q_2 < \cdots < p_{k+1} < q_{k+1}$ such that if $c_j \in \mathcal{I}(O_Y)$, then $p_i \leq j \leq q_i$ for some $i = 1, \ldots, k+1$. $\square$

Let $\varepsilon \colon \mathsf{Acyc}(Y) \longrightarrow \mathsf{Acyc}(Y')$ be the restriction map that sends O_Y to $O_{Y'}$. Clearly, this map extends to a map $\varepsilon^* \colon \mathsf{Acyc}(Y)/{\sim_\kappa} \longrightarrow \mathsf{Acyc}(Y')/{\sim_\kappa}$. Define

$$\mathcal{I}^*_e \colon \mathsf{Acyc}(Y')/{\sim_\kappa} \longrightarrow \mathsf{Acyc}_{\leq}(Y)$$

by $\mathcal{I}^*_e([O_{Y'}]) = \mathcal{I}(O^1_Y)$ for any $O^1_Y \in [O_Y]$ such that $\varepsilon^*([O_Y]) = [O_{Y'}]$ with $|\mathcal{I}(O^1_Y)| \geq 3$, and $\mathcal{I}^*_e([O_{Y'}]) = \{v, w\}$ if no such acyclic orientation O^1_Y exists.

Proposition 3.27. *The map $\mathcal{I}_e^*$ is well-defined, and the diagram*

$$\begin{tikzcd}
\mathsf{Acyc}(Y)/{\sim_\kappa} \arrow[r, "\mathcal{I}^*"] \arrow[d, "\varepsilon^*"'] & \mathsf{Acyc}_{\leq}(Y) \\
\mathsf{Acyc}(Y')/{\sim_\kappa} \arrow[ur, dashed, "\mathcal{I}_e^*"'] &
\end{tikzcd}$$

commutes.

Proof. Let $[O_{Y'}] \in \mathsf{Acyc}(Y')/{\sim_\kappa}$. If there is at most one orientation $O_Y \in \mathsf{Acyc}(Y)$ such that $|\mathcal{I}(O_Y)| \geq 3$ and $\varepsilon(O_Y) \in [O_{Y'}]$, or if all orientations of the form O_Y^1 in the definition of $\mathcal{I}_e^*$ are κ-equivalent, then both statements of the proposition are clear. Assume therefore that there are acyclic orientations $O_Y^\pi, O_Y^\sigma \in \mathsf{Acyc}(Y)$ with $O_Y^\pi \nsim_\kappa O_Y^\sigma$, but $\eta_e^*([O_Y^\pi]) = \eta_e^*([O_Y^\sigma])$ and $|\mathcal{I}(O_Y^\pi)|, |\mathcal{I}(O_Y^\sigma)| \geq 3$. It suffices to prove that in this case,

$$\mathcal{I}(O_Y^\pi) = \mathcal{I}(O_Y^\sigma)\ . \tag{3.9}$$

This is equivalent to showing that the set of vw-paths (directed paths from v to w) in $O_{Y'}^\pi$ is the same as the set of vw-paths in $O_{Y'}^\sigma$. From this it will also follow that the diagram commutes. By assumption, both of these orientations contain at least one vw-path. We will consider separately the cases when these orientations share or do not share a common vw-path.

Case 1: $O_{Y'}^\pi$ and $O_{Y'}^\sigma$ share no common vw-path. Let P_1 be a length-k_1 vw-path in $O_{Y'}^\pi$, and let P_2 be a length-k_2 vw-path in $O_{Y'}^\sigma$. Suppose that in $O_{Y'}^\pi$ there are k_2^+ edges along P_2 oriented from v to w, and k_2^- edges oriented from w to v. Likewise, suppose that in $O_{Y'}^\sigma$ there are k_1^+ edges along P_1 oriented from v to w, and k_1^- edges oriented from w to v. If $C = P_1 P_2^{-1}$

(the cycle formed by traversing P_1 followed by P_2 in reverse), then

$$\nu_C(O^{\pi}_{Y'}) = k_1^+ + k_1^- + k_2^- - k_2^+, \qquad \nu_C(O^{\sigma}_{Y'}) = k_1^+ - k_1^- - k_2^- - k_2^+ \ .$$

Equating these values yields $k_1^- + k_2^- = 0$, and since these are non-negative integers, $k_1^- = k_2^- = 0$. We conclude that P_1 is a vw-path in $O^{\sigma}_{Y'}$ and P_2 is a vw-path in $O^{\pi}_{Y'}$, contradicting the assumption that $O^{\pi}_{Y'}$ and $O^{\sigma}_{Y'}$ share no common vw-paths.

Case 2: $O^{\pi}_{Y'}$ and $O^{\sigma}_{Y'}$ share a common vw-path P_1, say of length k_1. If these are the only vw-paths, we are done. Otherwise, assume without loss of generality that P_2 is another vw-path in $O^{\pi}_{Y'}$, say of length k_2. Then if $C = P_1 P_2^{-1}$, we have $\nu_C(O^{\pi}_{Y'}) = k_1 - k_2$, and hence $\nu_C(O^{\sigma}_{Y'}) = k_1 - k_2$. Therefore, P_2 is a vw-path in $O^{\sigma}_{Y'}$ as well. Because P_2 was arbitrary, we conclude that $O^{\pi}_{Y'}$ and $O^{\sigma}_{Y'}$ share the same set of vw-paths. Since Case 1 is impossible, we have established (3.9), and the proof is complete. $\square$

Let $O_Y \in \mathsf{Acyc}(Y)$ and assume $I = \mathcal{I}(O_Y)$ has at least two vertices. We write Y_I for the graph formed from Y by contracting all vertices in I to a single vertex denoted V_I. If I only contains v and w then $Y_I = Y''_e$. Moreover, O_Y gives rise to an orientation O_{Y_I} of Y_I, and this orientation is clearly acyclic.

Proposition 3.28. *Let $O^1_Y, O^2_Y \in \mathsf{Acyc}(Y)$ and assume $\mathcal{I}(O^1_Y) = \mathcal{I}(O^2_Y)$. If $O^1_Y \nsim_\kappa O^2_Y$ then $[O^1_{Y_I}] \nsim_\kappa [O^2_{Y_I}]$.*

Proof. We prove the contrapositive statement. Set $I = \mathcal{I}(O^1_Y)$, suppose $|I| = k$, and let $v_1 v_2 \cdots v_k$ be a linear extension of I. For any click-sequence $\mathbf{c}_I$ between two acyclic orientations $O^1_{Y_I}$ and $O^2_{Y_I}$ in $\mathsf{Acyc}(Y_I)$, let $\mathbf{c}$ be the click-sequence formed by replacing every occurrence of $c_i = V_I$ in $\mathbf{c}_I$ by the sequence $v_1 \cdots v_k$. Then $\mathbf{c}(O_Y^1) = O_Y^2$ and $O_Y^1 \sim_\kappa O_Y^2$ as claimed. $\square$

3.8. **A recurrence for $\kappa(Y)$.** In this section, we will utilize the results in the previous section to establish a bijection from $\mathsf{Acyc}(Y)/\sim_\kappa$ to the disjoint union $\big(\mathsf{Acyc}(Y'_e)/\sim_\kappa \cup \mathsf{Acyc}(Y''_e)/\sim_\kappa\big)$ for any cycle-edge e, which will in turn imply the following theorem:

Theorem 3.29. *Let Y be an undirected graph and let e be a cycle-edge. Then κ satisfies the recurrence relation*

$$\kappa(Y) = \kappa(Y'_e) + \kappa(Y''_e)\ . \tag{3.10}$$

For a κ-class $[O_Y]$, let O_Y^π denote an orientation in $[O_Y]$ such that $\pi = v\pi_2 \cdots \pi_n$ and $w = \pi_i$ for i minimal. Define the map

$$\Theta\colon \mathsf{Acyc}(Y)/\sim_\kappa \longrightarrow \big(\mathsf{Acyc}(Y')/\sim_\kappa\big) \bigcup \big(\mathsf{Acyc}(Y'')/\sim_\kappa\big) \tag{3.11}$$

by

$$\Theta\colon [O_Y] \longmapsto \begin{cases} [O_{Y''}^\pi], & \exists O_Y^\pi \in [O_Y] \text{ with } \pi = vw\pi_3 \cdots \pi_n \\ [O_{Y'}^\pi], & \text{otherwise.} \end{cases} \tag{3.12}$$

Note that $[O_Y]$ is mapped into $\mathsf{Acyc}(Y'')/\sim_\kappa$ if and only if the only vertices in $\mathcal{I}_e^*([O_Y])$ are v and w. Since κ-equivalence over Y implies κ-equivalence over Y', Θ does not depend on the choice of O_Y^π, and is thus well-defined. The results we have derived for the vw-interval now allow us to establish the following:

Theorem 3.30. *The map Θ is a bijection.*

Proof. We first prove that Θ is surjective. Let $I = \{v, w\}$ and consider an element $[O_{Y''}] \in \mathsf{Acyc}(Y'')/\sim_\kappa$ with $O^\pi_{Y''} \in [O_{Y''}]$ where $\pi = V_I\pi_2 \cdots \pi_{n-1}$. Let $\pi^+ = vw\pi_2 \cdots \pi_{n-1} \in S_Y$. Clearly $[O^{\pi^+}_Y] \in \mathsf{Acyc}(Y)/\sim_\kappa$ is mapped to $[O_{Y''}]$ by Θ.

Next, consider an element $[O_{Y'}] \in \mathsf{Acyc}(Y')/\sim_\kappa$. If there is no element $O^\pi_{Y'}$ of $[O_{Y'}]$ such that $\pi = vw\pi_3 \cdots \pi_n$, then no elements of $[O_Y]$ are of this form either, and by definition $[O_{Y'}]$ has a preimage under Θ. We are left with the case where $[O_{Y'}]$ contains an element $O^\pi_{Y'}$ such that $\pi = vw\pi_3 \cdots \pi_n$, and we must show that there exists $O^{\pi'}_{Y'} \in [O_{Y'}]$ such that $[O^{\pi'}_Y]$ contains no element of the form O^σ_Y with $\sigma = vw\sigma_3 \cdots \sigma_n$. Note that if $\sigma = vw\sigma_3 \cdots \sigma_n$, then the vertices in $\mathcal{I}(O^\sigma_Y)$ are precisely v and w. If the orientation $O_{Y'}$ had a directed path from v to w, then the corresponding orientation $O_Y \in \mathsf{Acyc}(Y)$ formed by adding the edge e with orientation (v, w) has vw-interval of size at least 3, so by Proposition 3.22, the acyclic orientation O_Y cannot be κ-equivalent to any orientation O^σ_Y such that $\sigma = vw\sigma_3 \cdots \sigma_n$.

Thus it remains to consider the case when $[O_{Y'}]$ contains no acyclic orientation with a directed path from v to w. Pick any simple undirected path P' from v to w in Y', which exists because e is a cycle-edge. Choose an orientation in $[O_{Y'}]$ for which $\nu_{P'}$ is maximal. Without loss of generality we may assume that $O_{Y'}$ is this orientation. be the orientation that agrees with $O_{Y'}$, and with e oriented as (w, v). Since we have assumed that there is no directed path from v to w this orientation is acyclic. We claim that for any $\sigma = vw\sigma_3 \cdots \sigma_n$ one has $O_Y^\sigma \notin [O_Y]$. To see this, assume the statement is false. Let P be the undirected cycle in Y formed by adding the edge e to the path P'. Because e is oriented as (v, w) in O_Y^σ and as (w, v) in O_Y, we have $\nu_P(O_Y^\sigma) = \nu_{P'}(O_{Y'}^\sigma) - 1$ and $\nu_P(O_Y) = \nu_{P'}(O_{Y'}) + 1$. If O_Y and O_Y^σ were κ-equivalent, then

$$\nu_{P'}(O_{Y'}^\sigma) - 1 = \nu_P(O_Y^\sigma) = \nu_P(O_Y) = \nu_{P'}(O_Y) + 1 ,$$

and thus $\nu_{P'}(O_{Y'}^\sigma) = \nu_{P'}(O_Y) + 2$. Any click-sequence mapping O_Y to O_Y^σ is a click-sequence from $O_{Y'}$ to $O_{Y'}^\sigma$. Therefore, $O_{Y'}^\sigma \in [O_{Y'}]$, which contradicts the maximality of $\nu_{P'}(O_{Y'})$. We therefore conclude that $O_Y^\sigma \notin [O_Y]$, that $\Theta([O_Y]) = [O_{Y'}]$, and hence that Θ is surjective.

We next prove that Θ is an injection. By Proposition 3.28 (with $I = \{v, w\}$), Θ is injective when restricted to the preimage of $[O_{Y''}]$ under Θ. Thus it suffices to show that every element in $\mathsf{Acyc}(Y')/\sim_\kappa$ has a unique preimage under Θ. By Proposition 3.27, every preimage of $[O_{Y'}]$ must have the same vw-interval I, containing $k > 2$ vertices. Suppose there were preimages $[O_Y^\pi] \neq [O_Y^\sigma]$ of $[O_{Y'}]$. By Proposition 3.28, it follows that $O_{Y_I}^\pi \nsim_\kappa O_{Y_I}^\sigma$. We will now show that this leads to a contradiction.

Assume that $\mathbf{c} = c_1 \cdots c_m$ is a click-sequence from $O_{Y'}^\pi$ to $O_{Y'}^\sigma$. If one of π or σ is not κ-equivalent to a permutation with vertices v and w in succession, then their corresponding κ-classes would be unchanged by the removal of edge e. In light of this, we may assume that $\pi = v\pi_2 \ldots \pi_{n-1}w$ and $\sigma = v\sigma_2 \ldots \sigma_{n-1}w$, and thus that $c_1 = v$ and $c_m = w$. By Proposition 3.25, we may assume that the vertices in I appear in $\mathbf{c}$ in some number of disjoint consecutive "blocks," i.e., subsequences of the form $c_i \cdots c_{i+k-1}$.

Replacing each of these blocks with V_I yields a click-sequence from $O^{\pi}_{Y_I}$ to $O^{\sigma}_{Y_I}$, contradicting the fact that $O^{\pi}_{Y_I} \nsim_{\kappa} O^{\sigma}_{Y_I}$. Therefore, no such click sequence $\mathbf{c}$ exists, and Θ must be an injection, and the proof is complete. $\square$

Clearly, Theorem 3.30 implies Theorem 3.29. It is also interesting to note that the bijection $\beta_e \colon \mathsf{Acyc}(Y) \longrightarrow \mathsf{Acyc}(Y'_e) \cup \mathsf{Acyc}(Y''_e)$ in (2.3) does *not* extend to a well-defined map on κ-classes.

3.9. **The Tutte polynomial.** In this section we relate the problem of computing $|\mathsf{Acyc}(Y)/\sim_{\kappa}|$ to two other enumeration problems where the recurrence in Theorem 3.29 holds. We will show how these problems are equivalent, and additionally, how they all can be computed through an evaluation of the Tutte polynomial. As a corollary we obtain a transversal of $\mathsf{Acyc}(Y)/\sim_{\kappa}$.

In [12] the notion of *cut-equivalence* of acyclic orientations is studied. A *cut* of a graph Y is a partition of the vertex set into two classes $\mathrm{v}[Y] = V_1 \sqcup V_2$, and where $[V_1, V_2]$ is the set of edges between V_1 and V_2. A cut of a graph Y is *oriented* with respect to $O_Y \in \mathsf{Acyc}(Y)$ if the edges of $[V_1, V_2]$ are all directed from V_1 to V_2, or are all directed from V_2 to V_1.

Definition 3.31 (Cut-equivalence)**.** Two acyclic orientations O_Y and O'_Y are *cut-equivalent* if the set $\{e \in \mathrm{e}[Y] \mid O_Y(e) \neq O'_Y(e)\}$ is (i) empty or is (ii) an oriented cut with respect to either O_Y or O'_Y.

The study of cut-equivalence in [12] was done outside the setting of Coxeter theory and SDSs, and here we provide the connection.

Proposition 3.32. *Two acyclic orientations of Y are κ-equivalent if and only if they are cut-equivalent.*

Proof. Suppose distinct elements O_Y and O'_Y in $\mathsf{Acyc}(Y)$ are cut-equivalent, and without loss of generality, that all edges of $[V_1, V_2]$ are oriented from V_1 to V_2 in O_Y. A click-sequence containing each vertex of V_1 precisely once maps O_Y to O'_Y, thus $O_Y \sim_\kappa O'_Y$.

Conversely, suppose that $O_Y \sim_\kappa O'_Y$, where O'_Y is obtained from O_Y by a click-sequence containing a single vertex v. Then O_Y and O'_Y are cut-equivalent, with the partition being $\{v\} \sqcup \mathrm{v}[Y] \setminus \{v\}$. $\square$

Obviously, the recurrence relation in (3.10) holds for the enumeration of both cut-equivalence and κ-equivalence classes.

Definition 3.33 (Tutte polynomial)**.** The *Tutte polynomial* of an undirected graph Y is defined recursively as follows. If Y has b bridges, ℓ loops, and no cycle-edges, then $T_Y(x,y) = x^b y^\ell$. If e is a cycle-edge of Y, then

$$T_Y(x,y) = T_{Y'_e}(x,y) + T_{Y''_e}(x,y) \ .$$

We remark that it is well-known that the number of acyclic orientations of a graph Y is $\alpha(Y) = T_Y(2,0)$. It was shown in [12] that the number of cut-equivalence classes can be computed through an evaluation of the Tutte polynomial as $T_Y(1,0)$, and thus $\kappa(Y) = T_Y(1,0)$. Some of the results we proved about the structure of $C(Y)$ and $D(Y)$ have a natural interpretation in the language of the Tutte polynomial. For example, Corollary 3.13 tells us that a connected graph Y is bipartite if and only if $T_Y(1,0)$ is odd. Corollary 3.10 implies that $n \cdot T_Y(1,0) \leq T_Y(2,0)$.

It is known that $T_Y(1,0)$ counts several other quantities, some of which can be found in [17]. One of these is $|\mathsf{Acyc}_v(Y)|$, the number of acyclic orientations of Y where a fixed vertex v is the unique source. As the next result shows, there is a bijection between $\mathsf{Acyc}(Y)/\sim_\kappa$ and $\mathsf{Acyc}_v(Y)$.

Proposition 3.34. *Let Y be a connected graph. For any fixed $v \in \mathrm{v}[Y]$, there is a bijection*

$$\phi_v \colon \mathsf{Acyc}_v(Y) \longrightarrow \mathsf{Acyc}(Y)/\sim_\kappa \ .$$

Proof. Since $\kappa(Y) = |\mathsf{Acyc}(Y)/\sim_\kappa| = T_Y(1,0) = |\mathsf{Acyc}_v(Y)|$ it is sufficient to show that there is a surjection $\phi_v \colon \mathsf{Acyc}_v(Y) \to \mathsf{Acyc}(Y)/\sim_\kappa$.

We first prove that each $A \in \mathsf{Acyc}(Y)/\sim_\kappa$ contains at least one element of $\mathsf{Acyc}_v(Y)$ by contradiction. Assume that $A \in \mathsf{Acyc}(Y)/\sim_\kappa$ contains no element of $\mathsf{Acyc}_v(Y)$, and choose $O_Y \in A$ such that v is a source. Clearly, the assumption implies that there exists infinite length click-sequences from O_Y not containing v. Let c be such a click-sequence, and let $V' \subset \mathrm{v}[Y]$ be the set containing all vertices that occur infinitely often in c. Then $V' \neq \emptyset$, and since $v \notin V'$ we have $\mathrm{v}[Y] \setminus V' \neq \emptyset$. Clearly, for such a click-sequence c to exist there can be no edges of the form $\{s,t\} \in \mathrm{e}[Y]$ with $s \in V'$ and $t \in \mathrm{v}[Y] \setminus V'$, and we are forced to conclude that Y is not connected, a contradiction. Hence any $A \in \mathsf{Acyc}(Y)/\sim_\kappa$ contains at least one element of $\mathsf{Acyc}_v(Y)$.

Clearly, non-equivalent κ-classes of Y cannot have elements of $\mathsf{Acyc}_v(Y)$ in common, and since $|\mathsf{Acyc}(Y)/\sim_\kappa| = |\mathsf{Acyc}_v(Y)|$ we conclude that each κ-class of Y contains a unique element of $\mathsf{Acyc}_v(Y)$. The map $\phi_v \colon \mathsf{Acyc}_v(Y) \to$

$\mathsf{Acyc}(Y)/{\sim_\kappa}$ defined by $\phi_v(O_Y) = [O_Y]$ is therefore a surjection, and by the initial comment, a bijection. $\square$

From Proposition 3.34 we immediately obtain:

Corollary 3.35. *For any vertex v of Y the set $\mathsf{Acyc}_v(Y)$ is a transversal of $\mathsf{Acyc}(Y)/{\sim_\kappa}$.*

In light of the results in this section, the recurrence in (3.10) may also be proven by showing that the map ϕ_v is injective. However, our proof offers insight into the structure of the κ-classes, and it is our hope that this may lead to new techniques for studying conjugacy classes of Coxeter groups.

3.10. **Examples.** The recurrence for $\kappa(Y)$, along with the fact that $\kappa(Y) = 1$ for a forest allows us to easily compute $\kappa(Y)$ for some common graph classes. First, we derive an explicit formula for graphs that are vertex joins.

Definition 3.36 (Vertex join)**.** The *vertex join* of a graph Y denoted $Y \oplus v$, is the graph

$$\mathrm{v}[Y \oplus v] = \mathrm{v}[Y] \sqcup \{v\}\ , \qquad \mathrm{e}[Y \oplus v] = \mathrm{e}[Y] \cup \big\{\{v, v'\} \mid v' \in \mathrm{v}[Y]\big\}\ .$$

In general, the recursion relation is unhelpful for computing $\kappa(Y \oplus v)$. However, it follows easily from Corollary 3.35.

Proposition 3.37. *If Y is a graph with $\mathrm{e}[Y] \neq \emptyset$, then*

$$\kappa(Y \oplus v) = 2\delta(Y \oplus v) = \alpha(Y)\ . \tag{3.13}$$

Proof. If v is a source of $Y \oplus v$, then it must be the unique source, and so $\kappa(Y \oplus v) = \alpha(Y)$ is immediate from Corollary 3.35. Since $\mathrm{e}[Y] \neq \emptyset$, the vertex join $Y \oplus v$ contains a cycle of length 3, is thus not bipartite, and so by Proposition 3.12 we have $2\delta(Y \oplus v) = \kappa(Y \oplus v)$. □

Proposition 3.37 allows us to compute κ for the complete graph K_n and the graph Wheel_n, the vertex join of Circ_n.

Corollary 3.38. *For $n \geq 3$, $\kappa(K_n) = (n-1)!$, $\delta(K_n) = (n-1)!/2$, $\kappa(\mathsf{Wheel}_n) = 2^n - 2$ and $\delta(\mathsf{Wheel}_n) = 2^{n-1} - 1$.*

Proof. There are $2^{\binom{n}{2}}$ orientations of K_n, and by the bijection in (2.1), precisely $\alpha(K_n)$ of these are acyclic, and this is equal to the number of components of the update graph $U(K_n)$. Since $U(K_n)$ consists of the $n!$ singleton vertices in S_Y, $\alpha(K_n) = n!$. By Proposition 3.37, $\kappa(K_n) = \alpha(K_{n-1}) = (n-1)!$. There are 2^n orientations of Circ_n, and all but two of them are acyclic. Therefore, $\kappa(\mathsf{Wheel}_n) = \alpha(\mathsf{Circ}_n) = 2^n - 2$. The corresponding values for δ are immediate from Proposition 3.37. □

From the recurrence relation, the value $\kappa(Y)$ for the circle graphs is immediate. We note that this was done explicitly in [37].

Corollary 3.39. $\kappa(\mathsf{Circ}_n) = n - 1$ *and* $\delta(\mathsf{Circ}_n) = \lceil \frac{n-1}{2} \rceil$.

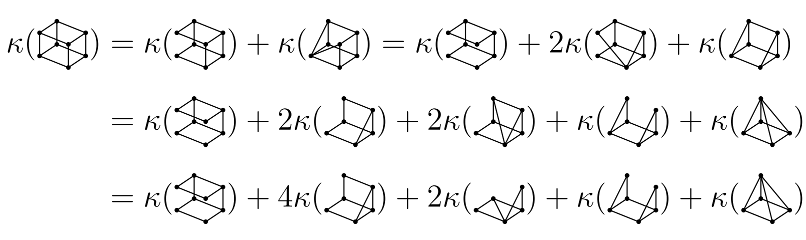
3.11. **Actions of $\mathsf{Aut}(Y)$.** In the case of functional equivalence, the action of $\mathsf{Aut}(Y)$ on the set $S_Y/{\sim_Y}$ gives rise to the weaker notion of dynamical equivalence. In the setting of cycle equivalence, we may identify elements of $S_Y/{\sim_\kappa}$ (or $S_Y/{\sim_\delta}$) that contain update orders related by some $\varphi \in \mathsf{Aut}(Y)$, as long as the functions $\mathfrak{F}_Y$ are $\mathsf{Aut}(Y)$-invariant. Two SDS maps arising from these update orders will be conjugate when restricted to the periodic points, i.e.,

$$\left(\varphi \circ [\mathfrak{F}_Y, \pi] \circ \varphi^{-1}\right)\big|_P = [\mathfrak{F}_Y, \varphi * \pi]\big|_P . \tag{3.14}$$

Even if π and $\varphi * \pi$ are not κ- or δ-equivalent, for any $\varphi \in \mathsf{Aut}(Y)$, the SDS maps $[\mathfrak{F}_Y, \pi]$ and $[\mathfrak{F}_Y, \varphi * \pi]$ will be dynamically equivalent, and hence cycle equivalent. Thus as in (2.9), counting the number of orbits under this action gives a potentially better upper bound for the number of SDS maps up to cycle equivalence, obtainable by varying the update order and fixing $\mathfrak{F}_Y$. We shall denote these quantities by $\bar{\kappa}(Y)$ and $\bar{\delta}(Y)$, respectively.

Example 3.40. Let $Y = Q_2^3$ be the binary 3-cube, which has automorphism group isomorphic to $S_4 \times C_2$. It is shown in [7] that $\alpha(Q_2^3) = 1862$ and that $\bar{\alpha}(Q_2^3) = 54$. Thus, there are at most 1862 functionally nonequivalent permutation SDSs over Q_2^3 for a fixed sequence of vertex functions. Likewise, there are at most 54 dynamically nonequivalent $\mathsf{Aut}(Q_2^3)$-invariant permutation SDSs. It is known that the bound $\bar{\alpha}(Q_2^3)$ is sharp, since it is realized for SDSs induced by, e.g. the nor_4-function.

The number of cycle equivalence classes is bounded above by $\kappa(Q_2^3)$, and from the recurrence relation we get (with some foresight at each step)



$$= 27 + 64 + 16 + 12 + 14 = 133 \,,$$

Since Q_2^3 is bipartite we also derive $\delta(Q_2^3) = (133 + 1)/2 = 67$, and thus in the case of $K = \mathbb{F}_2$ there are at most 67 cycle classes for a fixed sequence of vertex functions. Straightforward (but somewhat lengthy) calculations show that $\bar{\kappa}(Q_2^3) = \bar{\delta}(Q_2^3) = 8$. In conclusion, we have:

$$\begin{aligned} &\alpha(Q_2^3) = 1862 , && \kappa(Q_2^3) = 133 , && \delta(Q_2^3) = 67 , \\ &\bar{\alpha}(Q_2^3) = 54 , && \bar{\kappa}(Q_2^3) = 8 , && \bar{\delta}(Q_2^3) = 8 . \end{aligned}$$

Thus if $\mathfrak{F}_Y$ is a sequence of $\mathsf{Aut}(Q_2^3)$-invariant Y-local functions, there are at most eight different periodic orbit configurations for permutation SDS maps $[\mathfrak{F}_Y, \pi]$ up to isomorphism. Moreover, because $\bar{\kappa}(Q_2^3) = \bar{\delta}(Q_2^3)$ taking vertex states from $K = \mathbb{F}_2$ does not improve this upper bound.

This example shows how when $\mathsf{Aut}(Y)$ is non-trivial, the functions $\kappa(Y)$ (and $\delta(Y)$ when $K = \mathbb{F}_2$) are not sharp upper bounds for the number of SDS maps up to cycle equivalence. However, it remains an open question if $\bar{\kappa}(Y)$ and $\bar{\delta}(Y)$ are sharp.

3.12. **Connections to node-firing games and representations of quivers.** Not surprisingly, acyclic orientations show up in other areas of mathematics, and we conclude this section by showing how the source-to-sink operations arise in these settings.

We have seen how $\alpha(Y)$ counts the number of chambers of the graphic hyperplane arrangement $\mathcal{H}(Y)$, so it is reasonable to expect that the source-to-sink operation should have a natural interpretation in this setting. In fact, the quantity $T_Y(1, 0)$ is the Möbius invariant of the intersection lattice of the graphic hyperplane arrangement of Y, and the κ-classes correspond to chambers bounded by certain linear functionals (see [31]).

The *chip-firing* game was introduced by Björner, Lovász, and Shor [11]. It is played over an undirected graph Y, and each vertex is given some number (possibly zero) of chips. If vertex i has degree d_i, and at least d_i chips on it, then a legal move (or a "click") of vertex i is a transfer of one chip to each neighboring vertex. This is in a sense a generalization of a source-to-sink operation, because to any acyclic orientation O_Y, an assignment to each vertex the number of chips equal to its out-degree gives a configuration where legal moves of the chip-firing game correspond to source-to-sink operations of O_Y. The chip-firing game is closely related to the *numbers game* [10]. In the numbers game over a graph Y, the legal sequences of moves are in 1–1 correspondence with the reduced words of the Coxeter group with Coxeter graph Y. For an excellent summary and comparison of these games, see [15]. We also point out a previous study of the Tutte polynomial in the context of the chip-firing game [26]. It is our hope that this paper will motivate the further pursuit of the connections between these topics, as well as closure to certain open problems in Coxeter theory, such as the sharpness of the bound $\kappa(Y)$ for the enumeration of conjugacy classes of Coxeter elements.

A quiver is a finite directed graph (loops and multiple edges are allowed), and appears primarily in the study of representation theory. A quiver Q with a field K gives rise to a path algebra KQ, and there is a natural correspondence between KQ-modules, and K-representations of Q. In fact, there is an equivalence between the categories of quiver representations, and modules over path algebras. A path algebra is finite-dimensional if and only if the quiver is acyclic, and the modules over finite-dimensional path algebras form a reflective subcategory. A *reflection functor* maps representations of a quiver Q to representations of a quiver Q', where Q' differs from Q by a source-to-sink operation [27]. We note that while the composition of n source-to-sink operations (one for each vertex) maps a quiver back to itself, the corresponding composition of reflection functors is not the identity, but rather a *Coxeter functor*. In fact, the same result in [39] about powers of Coxeter elements being reduced was proven previously using techniques from the representation theory of quivers [23].

4. Word-independence and dynamics groups

The central theme of this section is *word-independence*. A sequence of Y-local functions is said to be word-independent if the set of periodic points of an SDS map over the functions is independent of the update order. When this happens, the Y-local functions generate a group called the *dynamics group*. If the vertex states are from $K = \mathbb{F}_2$, then this group is the homomorphic image of a Coxeter group. The dynamics group describes how the periodic points are permuted by the local functions. We study some general properties of word-independent functions and dynamics groups, and then apply these techniques to a simple class of SDSs called elementary *asynchronous cellular automata,* or ACAs. These are defined over the circular graph Circ_n, and can be thought of as elementary finite cellular automata, where there vertex functions are updated asynchronously instead of in parallel. There are 256 local rules that give rise to ACAs, and we prove that precisely 104 of them are word-independent for all $n > 3$. This is a significant extension of a recent theorem in [18]. We then classify these 104 rules to better understand their dynamics, and to examine the sets of periodic points and the dynamics groups of these systems. In addition to the pure intellectual merit of this problem, it is a good starting point for the study of update-order stochastic SDSs. We conclude this section by outlining directions for future research.

4.1. **Word-independence.**

Definition 4.1 (ω-independence)**.** A sequence of Y-local functions $\mathfrak{F}_Y$ is called ω-*independent*, if $\mathsf{Per}[\mathfrak{F}_Y, \omega] = \mathsf{Per}[\mathfrak{F}_Y, \omega']$ for all fair update orders $\omega, \omega' \in W_Y$, and is called π-*independent* if $\mathsf{Per}[\mathfrak{F}_Y, \pi] = \mathsf{Per}[\mathfrak{F}_Y, \pi']$ for all simple update orders $\pi, \pi' \in S_Y$.

Every ω-independent $\mathfrak{F}_Y$ is trivially π-independent. The following proposition is crucial in showing that the converse holds as well.

Proposition 4.2. *If $\mathfrak{F}_Y$ is π-independent, and $P = \mathsf{Per}[\mathfrak{F}_Y, \pi]$, then each $F_i \colon P \to P$ is invertible.*

Proof. Suppose that $\mathfrak{F}_Y$ is π-independent. Let $\pi \in S_Y$, and let $\sigma = (\pi_2, \pi_3, \ldots, \pi_n, \pi_1)$. Observe that for all $k \in \mathbb{N}$, $F_{\pi_1} \circ [\mathfrak{F}_Y, \pi]^k = [\mathfrak{F}_Y, \sigma]^k \circ F_{\pi_1}$. Choose $k \in \mathbb{N}$ large enough so that $[\mathfrak{F}_Y, \pi]^k(K^n) = [\mathfrak{F}_Y, \sigma]^k(K^n) = P$. Then

$$F_{\pi_1}(P) = F_{\pi_1} \circ [\mathfrak{F}_Y, \pi]^k(K^n) = [\mathfrak{F}_Y, \sigma]^k \circ F_{\pi_1}(K^n) \subseteq P,$$

and it follows that

$$F_{\pi_1}(P) \subseteq P. \tag{4.1}$$

Moreover, we have $P = [\mathfrak{F}_Y, \pi](P)$ and $|P| = |[\mathfrak{F}_Y, \pi](P)| \leq |F_{\pi_1}(P)|$. Therefore, equality must hold in (4.1), and thus F_{π_1} is invertible on P. $\square$

Remark 4.3. The proof of Proposition 4.2 did not use the fact that π was a simple update order, and thus the same argument holds under the assumption that $\mathfrak{F}_Y$ is ω-independent.

In fact, it is straightforward to show that π- and ω-independence are equivalent conditions.

Corollary 4.4. *A sequence $\mathfrak{F}_Y$ of Y-local functions is ω-independent if and only if it is π-independent.*

Proof. Suppose $\mathfrak{F}_Y$ is π-independent. By Proposition 4.2, $\mathsf{Per}[\mathfrak{F}_Y, \omega] \supseteq \mathsf{Per}[\mathfrak{F}_Y, \pi]$ for any fair $\omega \in W_Y$ and $\pi \in S_Y$. For the reverse inclusion, observe that by Proposition 4.2 and Remark 4.3, each F_i is a bijection on $\mathsf{Per}[\mathfrak{F}_Y, \omega]$, and thus for any $\mathbf{y} \in \mathsf{Per}[\mathfrak{F}_Y, \omega]$, we have $[\mathfrak{F}_Y, \pi](\mathbf{y}) \subseteq \mathsf{Per}[\mathfrak{F}_Y, \pi]$. $\square$

In light of Corollary 4.4, we shall call ω- (or π-) independence simply *word-independence*. Even though word-independence is too strong to expect generally, there are several classes of SDS maps that have this property. It is fairly easy to show that both invertible and fixed point systems are word-independent.

Definition 4.5. A sequence of local functions $\mathfrak{F}_Y$ (and a corresponding SDS over $\mathfrak{F}_Y$) is a *fixed point system* if for every $\pi \in S_Y$, the SDS map $[\mathfrak{F}_Y, \pi]$ fixes every point in $\mathsf{Per}[\mathfrak{F}_Y, \pi]$.

Proposition 4.6 (Fixed point systems)**.** *Fixed point systems are word-independent.*

Proof. If $\mathbf{y}$ is fixed by $[\mathfrak{F}_Y, \pi]$, then because the local functions can only change one coordinate at a time, $\mathbf{y}$ must be fixed by each F_i in $\mathfrak{F}_Y$, in which case it is a fixed point of $[\mathfrak{F}_Y, \pi]$ for every $\pi \in S_Y$. Therefore, a point of K^n is fixed by $[\mathfrak{F}_Y, \pi]$ if and only if it is fixed by $[\mathfrak{F}_Y, \sigma]$ for every $\sigma \in S_Y$. Since by hypothesis, the only periodic points are fixed points, then every permutation SDS map has the same set of periodic points, hence $\mathfrak{F}_Y$ is π-independent, and by Corollary 4.4, word-independent as well. $\square$

It is essentially immediate to show that invertible functions are word-independent.

Proposition 4.7 (Invertible functions)**.** *If every local function F_i in $\mathfrak{F}_Y$ is a bijection, then for every update order $\omega \in W_Y$, $\mathsf{Per}[\mathfrak{F}_Y, \omega] = K^n$, and consequently, $\mathfrak{F}_Y$ is word-independent.*

Proof. Since every F_i is a bijection, so is the SDS map $[\mathfrak{F}_Y, \omega]$, thus $\mathsf{Per}[\mathfrak{F}_Y, \omega]$ $= K^n$ for every $\omega \in W_Y$. $\square$

Sometimes, we can prove that a particular non-invertible SDS is word-independent by showing that when restricted to its periodic point set, it agrees with an invertible SDS. We will use this technique later when proving that certain asynchronous cellular automata are word-independent. In fact, a more general statement holds.

Theorem 4.8. *Suppose that for all $\pi \in S_Y$, the periodic points of an* SDS *map $[\mathfrak{F}_Y, \pi]$ are all contained in a set $M \subseteq K^n$, and*

$$[\mathfrak{F}_Y, \pi](M) = M.$$

Then $\mathfrak{F}_Y$ is word-independent, and $\mathsf{Per}(\mathfrak{F}_Y) = M$.

Proof. By assumption,

$$\mathsf{Per}[\mathfrak{F}_Y, \pi] \subseteq M. \tag{4.2}$$

We will show the reverse inclusion by producing an injection $M \hookrightarrow \mathsf{Per}[\mathfrak{F}_Y, \pi]$. Since M is invariant under $[\mathfrak{F}_Y, \pi]$, the i^{th} iteration $[\mathfrak{F}_Y, \pi]^i(M) = M$ for each $i \in \mathbb{N}$. Moreover, for some $N \in \mathbb{N}$, if $i \geq N$, then

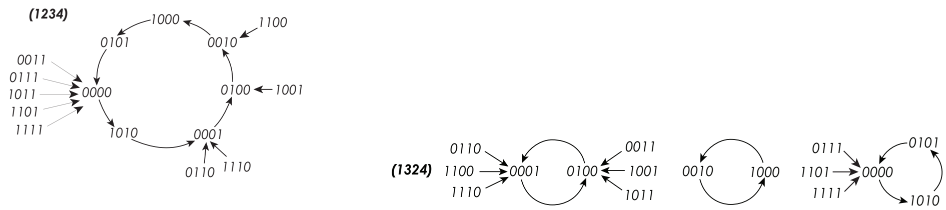
$$[\mathfrak{F}_Y, \pi]^i(M) \subseteq \mathsf{Per}[\mathfrak{F}_Y, \pi] \; .$$

We conclude that the mapping

$$[\mathfrak{F}_Y, \pi]^i : M \hookrightarrow \mathsf{Per}[\mathfrak{F}_Y, \pi]$$

is an injection, thus equality holds in (4.2). $\square$

This last theorem exemplifies the fact that word-independence is a property of the periodic point sets as a whole rather than the cycle structure within them. The periodic states of a word-independent SDS will typically have different cycle configurations under different update orders, as shown by the example in Figure 4.1. In the next section, we define the dynam-



(a) $\Gamma[\mathsf{Nor}_{\mathsf{Circ}_4},\ 1234]$ (b) $\Gamma[\mathsf{Nor}_{\mathsf{Circ}_4},\ 1324]$

FIGURE 4.1. Phase spaces of an SDS with different update orders. The cycle structure is different for the two systems, but the sets of periodic points are the same.

ics group, which helps us better understand how the local functions of a word-independent SDS permute the periodic points within these sets.

4.2. **Dynamics and Coxeter groups.** Proposition 4.2 ensures that for any word-independent SDS, the local functions permute the set of periodic points. Therefore, we may define the group of permutations of periodic points for any word-independent SDS.

Definition 4.9. Let $\mathfrak{F}_Y$ be word-independent, and let F_i^* and $[\mathfrak{F}_Y,\pi]^*$ denote the maps F_i and $[\mathfrak{F}_Y,\pi]$, restricted to $P = \mathsf{Per}(\mathfrak{F}_Y)$. If $W' \subseteq W_Y$ is a collection of update orders, then the group

$$H(\mathfrak{F}_Y, W') = \langle [\mathfrak{F}_Y,\omega]^*\colon \omega \in W' \rangle$$

is called the *dynamics group* of $\mathfrak{F}_Y$ and W'. Two special cases are of particular interest. The first, when $W' = \{i\}_{1\le i\le n}$, is

$$G(\mathfrak{F}_Y) := H(\mathfrak{F}_Y, W_Y) = \langle F_i^*\colon F_i \in \mathfrak{F}_Y \rangle\ , \tag{4.3}$$

and is called the *full dynamics group*, or just simply the dynamics group of $\mathfrak{F}_Y$. The second case, when $W' = S_Y$, is

$$H(\mathfrak{F}_Y) := H(\mathfrak{F}_Y, S_Y) = \langle [\mathfrak{F}_Y, \omega]^* \colon \omega \in S_Y \rangle \ , \tag{4.4}$$

and is called the *permutation dynamics group* of $\mathfrak{F}_Y$. When it is clear from the context what $\mathfrak{F}_Y$ is, we shall denote the groups in (4.3) and (4.4) by just G and H, respectively.

Let $U, V \subseteq W_Y$, and let U^* and V^* denote the respective Kleene closures (closure under string concatenation). It is clear that if $U^* \subseteq V^*$, then $H(\mathfrak{F}_Y, U) \leq H(\mathfrak{F}_Y, V)$.

Example 4.10. Consider the Y-local functions $\overline{\mathsf{Id}}_Y = (\overline{\mathsf{Id}}_i)_{i=1}^n$ induced by the vertex functions

$$\overline{\mathsf{id}} \colon \mathbb{F}_2^k \longrightarrow \mathbb{F}_2 \ , \qquad \overline{\mathsf{id}} \colon (y_1, \ldots, y_{i-1}, y_i, y_{i+1}, \ldots, y_n) \longmapsto \bar{y}_i \ .$$

Clearly, the dynamics of this system is independent of update order, and every SDS map $[\overline{\mathsf{Id}}_Y, \pi]$ has order 2, each one being the inversion map $\mathbf{y} \mapsto \bar{\mathbf{y}}$, regardless of π. Therefore, $H(\overline{\mathsf{Id}}_Y) \cong C_2$ and $G(\overline{\mathsf{Id}}_Y) \cong C_2^n$, where C_2 is the cyclic group of order 2. (We will continue to use C_k instead of Z_k for the cyclic group of order k, to remain consistent with our notation in Section 3.)

When $K = \mathbb{F}_2$, there is a connection between dynamics groups and Coxeter groups, which can be seen readily by setting

$$m_{ij} = |F_i^* \circ F_j^*| \ ,$$

that is, the order of $F_i^* \circ F_j^*$. By Proposition 3.3, $F_i \circ F_i$ is the identity function when restricted to $\mathsf{Per}(\mathfrak{F}_Y)$. Therefore, $m_{ii} = 1$. One difference from the relations of a Coxeter group is that in the presentation of the dynamics group, the relation $m_{ij} = 1$ is allowed for $i \neq j$. However, the next proposition describes exactly when this can happen.

Proposition 4.11. *When $i \neq j$, $m_{ij} = 1$ if and only if F_i^* and F_j^* are the identity functions on $\mathsf{Per}(\mathfrak{F}_Y)$.*

Proof. Clearly, if F_i and F_j fix all $\mathbf{y} \in \mathsf{Per}(\mathfrak{F}_Y)$, then $m_{ij} = 1$. Conversely, if $m_{ij} = 1$, then $F_i \circ F_j(\mathbf{y}) = \mathbf{y}$. Because F_i and F_j are Y-local functions, $F_i \circ F_j$ changes $\mathbf{y}$ by changing the j^{th}, and then the i^{th} coordinate. If $F_i \circ F_j(\mathbf{y}) = \mathbf{y}$, then $\mathbf{y}$ is neither changed by F_i nor F_j. Since this holds for all $\mathbf{y} \in \mathbb{F}_2^n$, F_i and F_j are the identity on $\mathsf{Per}(\mathfrak{F}_Y)$. $\square$

By Proposition 4.6, fixed point systems are word-independent. The following corollary shows that these are precisely the functions that have trivial dynamics group.

Corollary 4.12. *The following are equivalent:*

- *$\mathfrak{F}_Y$ is a fixed point system.*
- *$m_{ij} = 1$ for all i and j.*
- *$G(\mathfrak{F}_Y)$ is the trivial group.*

For any Coxeter group, the matrix (m_{ij}) is called the *Coxeter matrix*. We can similarly define such a matrix for a word-independent SDS and its dynamics group. Without loss of generality, we may assume that the vertices in Y are ordered so that for some $k \leq n$, the function F_i is not the identity on $\mathsf{Per}(\mathfrak{F}_Y)$ iff $i \leq k$. The number k is called the *rank* of the dynamics group. The trivial group (i.e., the dynamics group of a fixed point system) is the only dynamics group with rank 0. The Coxeter matrix of $\mathfrak{F}_Y$ is the $n \times n$ matrix

$$M(\mathfrak{F}_Y) = \begin{bmatrix} C(\mathfrak{F}_Y) & 2 \\ 2 & 1 \end{bmatrix} \tag{4.5}$$

in block form. Here, $C(\mathfrak{F}_Y)$ is a $k \times k$ matrix, there is an $(n-k) \times (n-k)$ block of 1s, and the remaining entries are 2s. Since $m_{ij} \geq 2$ for all distinct $i, j \leq k$, $C(\mathfrak{F}_Y)$ is the matrix of a Coxeter group. Hence there exists a homomorphism from a Coxeter group onto $G(\mathfrak{F}_Y)$:

$$\langle s_i, \ldots, s_k \mid (s_i s_j)^{m_{ij}} \rangle \longrightarrow G(\mathfrak{F}_Y)\ ,$$

defined by the mapping $s_i \mapsto F_i$. A curious problem regarding the dynamics group stems from the observation that in general, Coxeter groups are infinite, but the dynamics group, being a group of permutations of a finite set, is always finite. Therefore, the dynamics group can be presented with relations $(F_i \circ F_j)^{m_{ij}}$, and some additional relations. An interesting research question is to determine these relations from the functions and the underlying graph.

Later in this section, we will study asynchronous cellular automata, which are defined over circular graphs, Circ_n. The local functions that arise are not always invariant under $\mathsf{Aut}(\mathsf{Circ}_n) \cong D_n$, but as we shall see, they are invariant under the transitive subgroup C_n. As the next result shows, this greatly simplifies the possibilities for the Coxeter matrix of the dynamics group.

Proposition 4.13. *Suppose $\mathfrak{F}_Y$ is a word-independent sequence of functions that is invariant under a transitive subgroup $H \leq \mathsf{Aut}(Y)$. Then rank$(G(\mathfrak{F}_Y)) = n$ if and only if $\mathsf{Fix}(\mathfrak{F}_Y) \neq \mathsf{Per}(\mathfrak{F}_Y)$.*

Proof. If the rank of the dynamics group is n, then there are clearly non-fixed points in $\mathsf{Per}(\mathfrak{F}_Y)$.

Conversely, suppose $\mathsf{Per}(\mathfrak{F}_Y)$ contains a non-fixed point $\mathbf{y}$. Then for some k, F_k is not the identity on $\mathsf{Per}(\mathfrak{F}_Y)$. Pick a vertex $\ell \neq k$. Because H is transitive, there exists some $h \in H$ such that $h(k) = \ell$. By Proposition 2.9, $h \circ F_k \circ h^{-1} = F_\ell$. By assumption, F_k is not the identity, so neither is F_ℓ. Since ℓ was arbitrary, the result follows. □

4.3. **Asynchronous cellular automata.** We will now use the tools and ideas that we have developed about word-independent functions to better understand a class of SDSs called elementary asynchronous cellular automata.

4.3.1. *Preliminaries.* Some of the simplest classical cellular automata are the one-dimensional CAs known as *elementary cellular automata*, or "ECAs". In an elementary CA, every vertex has precisely two neighbors, the vertex states are from $\mathbb{F}_2$, and all local functions are the same (i.e., C_n-invariant). Since every vertex has two neighbors, the underlying graph is either an infinite line, or a circle, and its vertex functions are of the form $f_i \colon \mathbb{F}_2^3 \to \mathbb{F}_2$. There are $2^8 = 256$ such functions, known as *Wolfram rules*, or ECA rules, and thus 256 types of elementary cellular automata. Even in such a restrictive situation there are many interesting properties that can be observed about the dynamics. For the remainder of this section the underlying graph will be $Y = \mathsf{Circ}_n$, and thus we will refer to S_Y simply as S_n, and the set of *fair* words in W_Y as W_n.

Definition 4.14 (Wolfram rules)**.** Let $F_i\colon \mathbb{F}_2^n \to \mathbb{F}_2^n$ be a Circ_n-local function at i, and let $f_i\colon \mathbb{F}_2^3 \to \mathbb{F}$ be the corresponding vertex function. The domain of f_i is a triple of the form (y_{i-1}, y_i, y_{i+1}). Call this a *local state configuration* and view all subscripts modulo n. In order to completely specify the function F_i it is sufficient to list how the i^{th} coordinate is updated for each of the eight possible local state configurations. More specifically, let $f_i(y_{i-1}, y_i, y_{i+1}) = z_i$. The vertex function f_i, and the corresponding local function F_i, henceforth both referred to as a *Wolfram rule*, is completely described by the following table.

(4.6)

$y_{i-1}y_iy_{i+1}$	111	110	101	100	011	010	001	000
z_i	a_7	a_6	a_5	a_4	a_3	a_2	a_1	a_0

The $2^8 = 256$ possible Wolfram rules can be indexed by an 8-digit binary number $a_7a_6a_5a_4a_3a_2a_1a_0$, or by its decimal equivalent $k = \sum_{i=0}^{7} a_i 2^i$. There is thus one *Wolfram rule* k for each integer $0 \leq k \leq 255$. For each such n, k and i, let $\mathsf{Wolf}_i^{(k)}$ denote the Circ_n-local function $F_i\colon \mathbb{F}_2^n \to \mathbb{F}_2^n$ just defined, let $\mathsf{wolf}_i^{(k)}$ denote the corresponding vertex function $f_i\colon \mathbb{F}_2^3 \to \mathbb{F}_2$, and let $\mathfrak{Wolf}_n^{(k)}$ denote the sequence of local functions $(\mathsf{Wolf}_1^{(k)}, \mathsf{Wolf}_2^{(k)}, \ldots, \mathsf{Wolf}_n^{(k)})$. We say that Wolfram rule k is word-independent whenever $\mathfrak{Wolf}_n^{(k)}$ is word-independent for all $n > 3$.

For each update order ω there is an SDS $(\mathsf{Circ}_n, \mathfrak{Wolf}_n^{(k)}, \omega)$ that can be thought of as a classical elementary CA, but with the update functions applied asynchronously. For this reason, such SDSs are called *asynchronous cellular automata* or ACAs.

4.3.2. *Main theorem.* We now state our main result about word-independent ACAs.

Theorem 4.15. *There are exactly* 104 *word-independent Wolfram rules. More precisely,* $\mathfrak{Wolf}_n^{(k)}$ *is word-independent for all* $n > 3$ *iff* $k \in \{$*0, 1, 4, 5, 8, 9, 12, 13, 28, 29, 32, 40, 51, 54, 57, 60, 64, 65, 68, 69, 70, 71, 72, 73, 76, 77, 78, 79, 92, 93, 94, 95, 96, 99, 102, 105, 108, 109, 110, 111, 124, 125, 126, 127, 128, 129, 132, 133, 136, 137, 140, 141, 147, 150, 152, 153, 156, 157, 160, 164, 168, 172, 184, 188, 192, 193, 194, 195, 196, 197, 198, 199, 200, 201, 202, 204, 205, 206, 207, 216, 218, 220, 221, 222, 223, 224, 226, 228, 230, 232, 234, 235, 236, 237, 238, 239, 248, 249, 250, 251, 252, 253, 254, 255*$\}$.

The result of Theorem 4.15 may seem surprising, because it says that the set of word-independent ACAs is the same for all n. It is a significant generalization of the main result of [18], which is that that precisely 11 of the 16 *symmetric* Wolfram rules are word-independent over Circ_n for all $n > 3$. In addition to identifying a large class of word-independent ACAs, the proof of Theorem 4.15 provides insight into the dynamics of these systems at both periodic and transient states, and thus serves as a foundation for the future study of their properties in a stochastic setting. We conclude this section with two remarks about the role played by computer investigations of these systems.

Remark 4.16 (Unlisted numbers)*.* The "only if" portion of this theorem was established experimentally. For each $4 \leq n \leq 9$, and for each $0 \leq k \leq 255$, a program calculated the set $\mathsf{Per}[\mathfrak{Wolf}_n^{(k)}, \pi]$ for all π of a transversal for $S_Y/\sim_Y$ that was also generated by the program. The program calculated the set of states that were periodic for *some* update order, as well as the set of states that were periodic for *every* update order. The word-independent rules are precisely those where these two sets are equal. For each of the 152 values of k not listed above, these sets were unequal for each $n = 4, \ldots, 9$, leaving the remaining 104 rules as the only candidates with the potential to be word-independent for all $n > 3$. Moreover, since a counterexample for one value of n leads to similar counterexamples for all multiples of n, these 104 rules are also the only ones that have any possibility of being word-independent for all sufficiently large values of n.

Remark 4.17 (Computational guidance)*.* These early computer-aided investigations also had a major impact on the "if" portion of the proof. The computer results helped identify patterns among the 104 rules, and by listing the actual sets of periodic points, we were able to conjecture what the sets of periodic points of certain rules were. In many cases, proving these conjectures immediately implied that the given rule was word-independent.

4.3.3. *Wolfram Rule Notations.* Once we determined the 104 rules that were word-independent, it was natural to attempt to classify them and

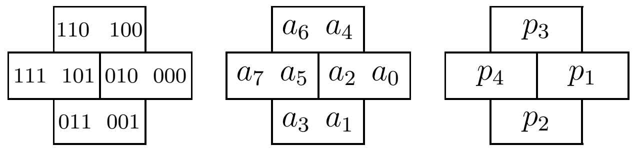


FIGURE 4.2. Grid notation for Wolfram rules

look for patterns about how they are are distributed among the 256 Wolfram rules. However, the conversion from binary to decimal obscures many structural details, so we introduce other ways to describe the Wolfram rules that makes their similarities and differences more immediately apparent, and significantly reduces the number of cases in the proof of Theorem 4.15.

Definition 4.18 (Grid notation)**.** For each binary $k = a_7a_6a_5a_4a_3a_2a_1a_0$ we arrange its digits in a grid. The 8 local state configurations naturally correspond to the vertices of the binary 3-cube Q_2^3. By projecting this into the plane, we get a grid, as in the left-hand side of Figure 4.2. In place of the local state configurations, we can place the corresponding binary digits of k, as shown in the center of Figure 4.2. The boxes have been added because the local state configurations come in pairs that differ only by the middle value. The arrangement of the binary digits of k in this manner is called the *grid notation* for k.

As an example, the grid notation for Wolfram rule $29 = 00011101$ is shown on the left-hand side of Figure 4.3. Because grid notation is sometimes cumbersome to work with we also define a very concise 4-symbol tag for each Wolfram rule.

Definition 4.19 (Tag)**.** The grid notation of a Wolfram rule consists of four pairs of binary digits, and there are four possibilities for each of these:

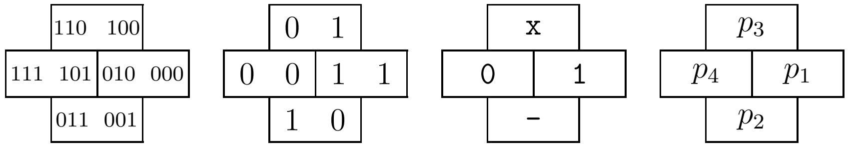


FIGURE 4.3. Converting Wolfram rule 29 = 00011101 into its tag `0x-1`.

(*i*) 00, (*ii*) 11, (*iii*) 01, and (*iv*) 10. These correspond to the image under rule k of the middle value of a pair of local state configurations. There are 4 possibilities: (*i*) both are mapped to 0, (*ii*) both are mapped to 1, (*iii*) both are changed, (*iv*) both are unchanged. We encode these 4 possibilities by the symbols `0`, `1`, `x`, and `-`, respectively. In other words '`0`' = $\boxed{0\quad 0}$, '`1`' = $\boxed{1\quad 1}$, '`x`' = $\boxed{0\quad 1}$, and '`-`' = $\boxed{1\quad 0}$. We label the symbols for the four boxes p_1, p_2, p_3 and p_4 as shown on the right-hand side of Figure 4.2 and define the *tag* of k to be the string $p_4p_3p_2p_1$. The numbering and the order of the p_is has been chosen to match the binary representation as closely as possible, with the hope of easing conversions between binary and tag representations. The process of converting Wolfram rule 29 to its tag `0x-1` is illustrated in Figure 4.3. Observe that a rule is invertible if and only if its tag contains no `0`s and no `1`s.

Definition 4.20 (Symmetric and asymmetric parts)**.** The middle row in the grid notation corresponds to the four symmetric local state configurations, and the top and bottom rows correspond to the four asymmetric local state configurations. Thus we call the middle row the *symmetric* portion of the grid and the top and bottom rows the *asymmetric* portion. With this in mind we call p_4p_1 the *symmetric part* of the tag $k = p_4p_3p_2p_1$ and p_3p_2 the *asymmetric part*.

Table 1 shows the 104 word-independent Wolfram rules listed in Theorem 4.15 arranged according to the symmetric and asymmetric parts of their tags. The rows list all 16 possibilities for the symmetric part of the tag while the columns list only 10 of the 16 possibilities for the asymmetric part since only these 10 occur among the 104 rules. Observe that the symmetric and asymmetric parts of the tag of rule k have a decimal equivalent whose sum is k. In this format the benefits of the tag representation should be clear. Far from being distributed haphazardly, the word-independent rules are clustered together in large blocks. Table 1 reveals a lot of structure, but some patterns remain slightly hidden due to the order in which the rows and columns are listed. For example, there is a 4×4 block of invertible rules obtained by restricting attention to the four rows that show up in the last column and the four columns that show up in the last row.

Proposition 4.21 (Invertible rules)**.** *Wolfram rules 51, 54, 57, 60, 99, 102, 105, 108, 147, 150, 153, 156, 195, 198, 201 and 204 are word-independent.*

Proof. These are the 16 invertible rules, because their tags contain no `0`s and `1`s. They are word-independent by Proposition 4.7. $\square$

4.3.4. *Dynamical equivalence.* In this section we utilize dynamical equivalence to reduce the proof of Theorem 4.15 to a more manageable size.

	p_3 p_2	- -	- 0	0 -	0 0	- 1	1 -	1 1	- x	x -	x x
p_4p_1		72	64	8	0	74	88	90	66	24	18
--	132	204	196	140	132	206	220	222	198	156	150
0-	4	76	68	12	4	78	92	94	70	28	
-0	128	200	192	136	128	202	216	218	194	152	
1-	164	236	228	172	164	238	252	254	230	188	
-1	133	205	197	141	133	207	221	223	199	157	
10	160	232	224	168	160	234	248	250	226	184	
01	5	77	69	13	5	79	93	95	71	29	
00	0	72	64	8	0						
x0	32		96	40	32						
0x	1	73	65	9	1						
-x	129	201	193	137	129				195	153	147
x-	36	108				110	124	126	102	60	54
x1	37	109				111	125	127			
1x	161					235	249	251			
11	165	237				239	253	255			
xx	33	105							99	57	51

TABLE 1. The 104 word-independent Wolfram rules arranged by the symmetric and asymmetric parts of their tags.

Proposition 4.22. *If two maps $\phi, \psi \colon K^n \to K^n$ are dynamically equivalent, related by $h \circ \phi = \psi \circ h$ for a surjection $h \colon K^n \to K^n$, then h is a bijection between the periodic points of ϕ and ψ.*

Proof. For each $m \in \mathbb{N}$, the diagram

$$\begin{tikzcd} K^n \arrow[r, "\phi^m"] \arrow[d, "h"'] & K^n \arrow[d, "h"] \\ K^n \arrow[r, "\psi^m"] & K^n \end{tikzcd}$$

commutes, where ϕ^m denotes the m^{th} iterate of ϕ. A state $\mathbf{y} \in K^n$ is periodic if and only if it is in the image of ϕ^m (or equivalently, $h \circ \phi^m$) for every $m \in \mathbb{N}$. Because the above diagram commutes, this is equivalent to $\mathbf{y}$ being in the image of $\psi^m \circ h$ (and hence ψ^m) for every m. Thus $h\colon \mathsf{Per}(\phi) \hookrightarrow \mathsf{Per}(\psi)$, and reversing the role of ϕ and ψ gives the opposite inclusion, and the result follows. $\square$

Corollary 4.23. *If $\mathfrak{F}_Y$ is word-independent, and $h \circ [\mathfrak{F}_Y, \omega] \circ h^{-1} = [\mathfrak{F}'_Y, \omega']$, then $\mathfrak{F}'_Y$ is word-independent as well.*

There are three basic ways to turn an ACA $(Y, \mathfrak{F}_Y, \omega)$ into a potentially different ACA $(Y, \mathfrak{F}'_Y, \omega')$, and these are best motivated by focusing on how to change the Wolfram rule vertex functions $f_i\colon \mathbb{F}_2^3 \to \mathbb{F}$. In particular, one can (i) reverse the role of left and right, (ii) reverse the role of 0 and 1, or (iii) do both of these. We call these alterations *reflection*, *inversion* and *reflection-inversion* of the rule, respectively. We note that in the case of reflection and reflection-inversion, there are multiple maps h that can achieve these alterations, namely the ones corresponding with different reflections in D_n. We will pick a convention and describe how our choice changes the local functions and the update order.

Definition 4.24 (Reflection maps)**.** The renumbering of the vertices we have in mind is achieved by the automorphism $r \in \mathsf{Aut}(\mathsf{Circ}_n) \cong D_n$ that sends vertex i to vertex $n+1-i$. This induces a map $R\colon \mathbb{F}_2^n \to \mathbb{F}_2^n$ by the action in (1.3), e.g., $\mathbf{y} = (y_1, y_2, \dots, y_n) = (y_n, \dots, y_2, y_1)$. We note that R is an involution. Additionally, r extends to a map $r\colon W_n \to W_n$ via the action in (1.4), e.g., $\omega = (\omega_1, \omega_2, \dots, \omega_m)$, then $r(\omega) = (r(\omega_1), r(\omega_2), \dots, r(\omega_m))$.

Definition 4.25 (Reflected rules)**.** If the vertices of Circ_n are renumbered via r, rule $\mathsf{Wolf}_i^{(k)}$ is applied, and then the renumbering is reversed, the net effect is the same as if a different Wolfram rule were applied to the vertex $r(i)$. Let ℓ be the number that represents this other Wolfram rule. The differences between k and ℓ are best seen in grid notation. The renumbering not only changes the vertex at which the rule seems to be applied, but it also reverses the order in which the 3 coordinates are listed in the restricted local form. Only the asymmetric local state configurations, i.e. the top and bottom rows of the grid, are altered by this change so that the grid for ℓ looks like a reflection of the grid for k across a horizontal line. We call ℓ the *reflection* of k and we define a map $\mathsf{refl}\colon \{0,\ldots,255\} \to \{0,\ldots,255\}$ with $\mathsf{refl}(k) = \ell$. On the level of tags, the only change is to switch order of p_2 and p_3, so, for example ℓ=`01-x` is the reflection of k=`0-1x`.

In short, when $\ell = \mathsf{refl}(k)$, $R \circ \mathsf{Wolf}_i^{(k)} \circ R = \mathsf{Wolf}_{r(i)}^{(\ell)}$ and, since R is an involution, this can be rewritten as $R \circ \mathsf{Wolf}_i^{(k)} = \mathsf{Wolf}_{r(i)}^{(\ell)} \circ R$.

Proposition 4.26. *If $\ell = \mathsf{refl}(k)$, then $\mathfrak{Wolf}_n^{(k)}$ is word-independent if and only if $\mathfrak{Wolf}_n^{(\ell)}$ is word-independent.*

Proof. Using the fact that $R \circ \mathsf{Wolf}_i^{(k)} = \mathsf{Wolf}_{r(i)}^{(\ell)} \circ R$, it follows immediately that

$$\begin{aligned} R \circ [\mathfrak{Wolf}_n^{(k)}, \omega] &= R \circ \mathsf{Wolf}_{\omega_m}^{(k)} \circ \cdots \circ \mathsf{Wolf}_{\omega_2}^{(k)} \circ \mathsf{Wolf}_{\omega_1}^{(k)} \\ &= \mathsf{Wolf}_{r(\omega_m)}^{(\ell)} \circ \cdots \circ \mathsf{Wolf}_{r(\omega_2)}^{(\ell)} \circ \mathsf{Wolf}_{r(\omega_1)}^{(\ell)} \circ R \\ &= [\mathfrak{Wolf}_n^{(\ell)}, r(\omega)] \circ R. \end{aligned}$$

By Corollary 4.23, word-independence of $\mathfrak{Wolf}_n^{(\ell)}$ implies word-independence of $\mathfrak{Wolf}_n^{(k)}$, and the converse holds because $\ell = \mathsf{refl}(k)$ implies $k = \mathsf{refl}(\ell)$. $\square$

We now show a similar result for the case of inversions.

Definition 4.27 (Inversion map)**.** Let $\mathbf{0}$ and $\mathbf{1}$ denote the constant states $(0, 0, \ldots, 0)$ and $(1, 1, \ldots, 1)$ in $\mathbb{F}_2^n$. Since the function $i(a) = 1 + a$ changes 0 to 1 and 1 to 0, the *inversion map* $I\colon \mathbb{F}_2^n \to \mathbb{F}_2^n$ sending $\mathbf{y}$ to $\mathbf{1} + \mathbf{y}$, has this effect on each coordinate of $\mathbf{y}$. The map I is an involution like R, and it is easily verified that they commute.

Definition 4.28 (Inverted rules)**.** If a state $\mathbf{y}$ is inverted, rule $\mathsf{Wolf}_i^{(k)}$ is applied, and then the inversion is reversed, the net effect is the same as if a different Wolfram rule were applied at vertex i, which again, we shall call ℓ. The differences between k and ℓ are again best seen in grid notation. The pre-inversion of states effects the local state configurations as though the grid had been rotated 180°. The second inversion merely changes every entry so that 1s becomes 0s and 0s become 1s. Thus the grid for ℓ can be obtained from the grid for k by rotating the grid and altering every entry. We call ℓ the *inversion* of k and define a map $\mathsf{inv}\colon \{0, \ldots, 255\} \to \{0, \ldots, 255\}$ with $\mathsf{inv}(k) = \ell$. On the level of tags, there are two changes that take place. Boxes p_1 and p_4 switch places as do boxes p_2 and p_3, but in the process, the boxes are turned over and the numbers changed. If we look at what this does to the entries in a box, 11 becomes 00, 00 becomes 11, while 10 and 01 are left unchanged. To formalize this, define a map $c\colon \{\mathtt{1}, \mathtt{0}, \mathtt{-}, \mathtt{x}\} \to \{\mathtt{1}, \mathtt{0}, \mathtt{-}, \mathtt{x}\}$ with $c(\mathtt{1}) = \mathtt{0}$, $c(\mathtt{0}) = \mathtt{1}$, $c(\mathtt{-}) = \mathtt{-}$, and $c(\mathtt{x}) = \mathtt{x}$. If k has tag $p_4p_3p_2p_1$, then ℓ has tag $c(p_1)c(p_2)c(p_3)c(p_4)$, so, for example, $\ell = \mathtt{x0-1}$ is the inversion of $k = \mathtt{0-1x}$.

In short, when $\ell = \mathsf{inv}(k)$, $I \circ \mathsf{Wolf}_i^{(k)} \circ I = \mathsf{Wolf}_i^{(\ell)}$ and, since I is an involution, this can be rewritten as $I \circ \mathsf{Wolf}_i^{(k)} = \mathsf{Wolf}_i^{(\ell)} \circ I$.

Proposition 4.29. *If $\ell = \mathsf{inv}(k)$, then $\mathfrak{Wolf}_n^{(k)}$ is word-independent if and only if $\mathfrak{Wolf}_n^{(\ell)}$ is word-independent.*

Proof. The value of ℓ was defined so that $I \circ \mathsf{Wolf}_i^{(k)} = \mathsf{Wolf}_i^{(\ell)} \circ I$, and thus $I \circ [\mathfrak{Wolf}_n^{(k)}, \omega] \circ I = [\mathfrak{Wolf}_n^{(\ell)}, \omega]$. As before, word-independence of $\mathfrak{Wolf}_n^{(k)}$ implies word-independence of $\mathfrak{Wolf}_n^{(\ell)}$ by Corollary 4.23, and the converse holds from the fact that $\ell = \mathsf{inv}(k)$ implies $k = \mathsf{inv}(\ell)$. $\square$

As an immediate corollary of Propositions 4.26 and 4.29, when $\ell = \mathsf{refl}(\mathsf{inv}(k)) = \mathsf{inv}(\mathsf{refl}(k))$, $\mathfrak{Wolf}_n^{(k)}$ is word-independent iff $\mathfrak{Wolf}_n^{(\ell)}$ is word-independent. If we partition the 256 Wolfram rules into equivalence classes of rules related by reflection, inversion or both, then there are 88 distinct equivalence classes. The 104 rules listed in Theorem 4.15 are contained in the union of 41 of them. These 88 classes are equivalently characterized as the orbits of the action of the Klein 4-group $\langle R, I \rangle$ on the set of 256 Wolfram rules. Table 2 displays representatives of these 41 classes in pared down versions of Table 1. We used reflection and inversion to eliminate 5 of the 10 columns. Every rule with a `1` in the asymmetric portion of its tag is the inversion of a rule with a `0` instead. In particular, the entries in the 3 columns headed `-1`, `1-` and `11` are inversions of the entries in the columns headed `0-`, `-0` and `00`, respectively. Next, since reflections switch p_2 and p_3 we can also eliminate the columns headed `-0`, `-x` as redundant. This leaves the 5 columns headed `00`, `0-`, `--`, `x-` and `xx`. Since the last 3 do not contain `0`s or `1`s, further inversions, or inversion-reflections can be used to identify redundant rows in these columns.

	p_3	0	0
	p_2	0	-
p_4p_1		0	8
--	132	132	140
0-	4	4	12
-0	128	128	136
00	0	0	8
-1	133	133	141
01	5	5	13
-x	129	129	137
0x	1	1	9
1-	164	164	172
10	160	160	168
x0	32	32	40

	p_3	-	x	x
	p_2	-	-	x
p_4p_1		72	24	18
--	132	204	156	150
x-	36	108	60	54
xx	33	105	57	51
-0	128	200	152	
10	160	232	184	
0-	4	76	28	
01	5	77	29	
00	0	72		
0x	1	73		

TABLE 2. The 41 word-independent Wolfram rules up to equivalence, separated into two tables by their behavior in asymmetric contexts.

As mentioned above, the 41 rules listed in Table 2 are representatives of the 41 distinct equivalence classes of rules whose word-independence needs to be established in order to prove Theorem 4.15. The rows in each table have been arranged to correspond as closely as possible with the structure of the proof. For example, the first three rows of the table on the right-hand side are the nine equivalence classes shown to be word-independent by Proposition 4.7.

4.3.5. *Potential functions.* In this section we prove that four large sets of Wolfram rules are word-independent. All of the proofs are similar and, when combined with Proposition 4.7, leave only 6 equivalence classes of Wolfram rules that need to be discussed separately. Most of the rules discussed in this section are fixed point systems, for which we will use the notion of a potential function. For the remaining rules, we will find an invertible rule that agrees with them on their periodic points sets, and then appeal to Proposition 4.2.

Definition 4.30 (Potential function)**.** A *non-increasing potential function* for $F\colon X \to X$ is a map $\rho\colon X \to \mathbb{R}$ such that $\rho(F(x)) \leq \rho(x)$ for all $x \in X$. A non-decreasing potential function is defined analogously. By *potential function*, we mean either a non-increasing or non-decreasing potential function.

A potential function narrows our search for periodic points since any element x with $\rho(F(x)) < \rho(x)$ cannot be periodic. A potential function of an SDS $(Y, \mathfrak{F}_Y, \omega)$ is a map $\rho\colon \mathbb{F}_2^n \to \mathbb{R}$ that is a potential function for the SDS map $[\mathfrak{F}_Y, \omega]$. The easiest way to create such a function is to find one that is a potential function for every local function F_i in $\mathfrak{F}_Y$. Naturally, ρ must be either a non-decreasing potential function for each F_i or a non-increasing potential function for each F_i, rather than a mixture of the two, for the inequalities to work out. When ρ has this stronger property we call it a *potential function for* $\mathfrak{F}_Y$ since such a ρ is a potential function for $(Y, \mathfrak{F}_Y, \omega)$ for every choice of update order ω.

The technique that we will utilize is to find an SDS potential function ρ such that for any periodic point $\mathbf{y}$, if $F_i(\mathbf{y}) \neq \mathbf{y}$, then $\rho(F_i(\mathbf{y})) \neq \rho(\mathbf{y})$. The existence of such a function ρ implies that all of the periodic points are fixed points, and thus is word-independence follows from Proposition 4.6.

Proposition 4.31. *Rules 0, 4, 8, 12, 72, 76, 128, 132, 136, 140 and 200 are word-independent.*

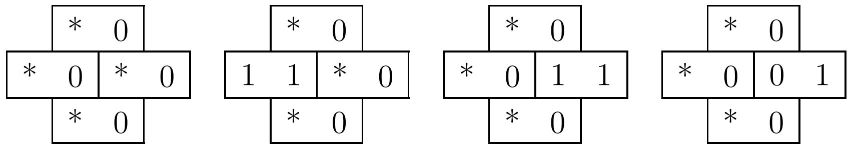


FIGURE 4.4. Four major classes of word-independent Wolfram rules.

Proof. If k is one of the numbers listed above, then its grid notation matches the leftmost form shown in Figure 4.4. (Each $*$ is a "wild-card", thus it can be either a 0 or a 1 so that 16 rules share this form, or 11 up to equivalence). The four 0s in the grid mean that local functions never remove 0s. Thus, the map ρ sending $\mathbf{y} \in \mathbb{F}_2^n$ to the number of 0s in $\mathbf{y}$ is a non-decreasing potential function for $\mathfrak{Wolf}_n^{(k)}$. Moreover, the local functions $\mathsf{Wolf}_i^{(k)}$ cannot change $\mathbf{y}$ without raising $\rho(\mathbf{y})$, so all periodic states are fixed points (for any update order), and by Proposition 4.6, $\mathfrak{Wolf}_n^{(k)}$ is word-independent. $\square$

For the next potential function, additional definitions are needed.

Definition 4.32 (Blocks)**.** A state $\mathbf{y} \in \mathbb{F}_2^n$ is thought of as a cyclic binary n-bit string with the indices taken modulo n, and a *substring* of $\mathbf{y}$ refers to a set of consecutive indices. We refer to maximal substrings of all 0s as *0-blocks* and maximal substrings of all 1s as *1-blocks.* If a block contains only a single number it is *isolated* and if it contains more than one number it is *non-isolated.*

As an example, the state $\mathbf{y} = 010110$ contains one isolated 0-block and one non-isolated 0-block of length 2 that wraps across the end of the word. We study how these blocks evolve as the local functions are applied. The decomposition of a Wolfram rule into its symmetric and asymmetric parts is particularly well adapted to the study of these evolutions. The asymmetric rules either make no change or shrink a non-isolated 1-block or 0-block from the left or the right, depending on which of the 4 asymmetric rules we are considering. Similarly, the 4 symmetric rules either do nothing, they remove an isolated block, or they create an isolated block in the interior of a long block.

Proposition 4.33. *Rules 160, 164, 168, 172 and 232 are word-independent.*

Proof. If k is one of the numbers listed above, then its grid notation matches the second form shown in Figure 4.4. The specified values (non wild-card) mean that (i) the only 0s ever removed are the isolated 0s and (ii) isolated 0s are never introduced. In particular, non-isolated blocks of 0s persist indefinitely, they might grow but they never shrink or split, and the isolated 0s, once removed, never return. Thus, the map ρ that sends $\mathbf{y}$ to the number of non-isolated 0s in $\mathbf{y}$ minus the number of isolated 0s in $\mathbf{y}$ is a non-decreasing potential function for $\mathfrak{Wolf}_n^{(k)}$. As before, the local functions $\mathsf{Wolf}_i^{(k)}$ cannot change $\mathbf{y}$ without raising $\rho(\mathbf{y})$, so all periodic states are fixed points, and by Proposition 4.6, $\mathfrak{Wolf}_n^{(k)}$ is word-independent. □

Proposition 4.34. *Rules 5, 13, 77, 133 and 141 are word-independent.*

Proof. If k is one of the numbers listed above, then its grid notation matches the third form shown in Figure 4.4. This time the specified values mean that (i) the only 0s that are removed create isolated 1s, and (ii) isolated 1s are never removed and they never stop being isolated. Thus the map ρ that sends $\mathbf{y}$ to the number of 0s in $\mathbf{y}$ plus *twice* the number of isolated

1s in $\mathbf{y}$ is a non-decreasing potential function for $\mathfrak{Wolf}_n^{(k)}$. Once again, the local functions $\mathsf{Wolf}_i^{(k)}$ cannot change $\mathbf{y}$ without raising $\rho(\mathbf{y})$, so all periodic states are fixed points, and by Proposition 4.6, $\mathfrak{Wolf}_n^{(k)}$ is word-independent. $\square$

The argument for the fourth collection is slightly more complicated.

Proposition 4.35. *Rules 1, 9, 73, 129 and 137 are word-independent.*

Proof. If k is one of the numbers listed above, then its grid notation matches the rightmost form shown in Figure 4.4. The specified values mean that (i) the only 0s that are removed create isolated 1s, but (ii) isolated 1s can also be removed. The map ρ that sends $\mathbf{y}$ to the number of 0s in $\mathbf{y}$ plus the number of isolated 1s in $\mathbf{y}$ is a non-decreasing potential function for $\mathsf{Wolf}_n^{(k)}$, but the difficulty is that there are local changes with $\rho(\mathsf{Wolf}_i^{(k)}(\mathbf{y})) = \rho(\mathbf{y})$. This is true for the local change $000 \to 010$ and for the local change $010 \to 000$. All other local changes raise the potential, but the existence of these two equalities indicates that there might be (and there are) states that are periodic under the action of some SDS map $[\mathfrak{Wolf}_n^{(k)}, \omega]$ without being fixed. Rather than appeal to a general theorem, we calculate its periodic states explicitly in this case.

Fix an update order $\omega \in W_n$ and, for convenience, let $F\colon \mathbb{F}_2^n \to \mathbb{F}_2^n$ denote the SDS map $[\mathfrak{Wolf}_n^{(k)}, \omega]\colon \mathbb{F}_2^n \to \mathbb{F}_2^n$. If $a_3 = 0$ and $\mathbf{y}$ contains a substring of the form 011, then $\rho(F(\mathbf{y})) > \rho(\mathbf{y})$ and $\mathbf{y}$ is not periodic under F. This is because either (i) the substring remains unaltered until its central coordinate is updated, at which point it changes to 0 and ρ is raised, or (ii) it is altered ahead of time by switching the 1 on the right to a

0 (also raising ρ), or by switching the 0 on the left to a 1 (impossible since $a_1 = a_5 = 0$). Analogous arguments show that if $a_6 = 0$ and $\mathbf{y}$ contains the substring 110, or if $a_7 = 0$ and $\mathbf{y}$ contains the substring 111, then $\mathbf{y}$ is not periodic under F. Let P be the subset of $\mathbb{F}_2^n$ where these situations do not occur. More specifically, if $a_3 = 0$ remove the states with 011 substrings, if $a_6 = 0$ remove the states with 110 substrings, and if $a_7 = 0$ remove the states with 111 substrings. If all three are equal to 1, then $P = \mathbb{F}_2^n$.

We claim that $P = \mathsf{Per}[\mathfrak{Wolf}_n^{(k)}, \omega]$, independent of the choice of ω. We have already shown $P \subset \mathsf{Per}[\mathfrak{Wolf}_n^{(k)}, \omega]$. Note that P is invariant under F (in the sense that $F(P) \subset P$) since the allowed local changes are not able to create the forbidden substrings when they do not already exist. Moreover, F restricted to P agrees with rule 201 = `---x`, the rule of this form with $a_3 = a_6 = a_7 = 1$, since whenever a_3, a_6 or a_7 is 0, P has been suitably restricted to make this fact irrelevant. Finally, rule 201 is invertible, thus F is injective on P, F permutes the states in P and a sufficiently high power of F is the identity, showing every state in P is periodic independent of our choice of ω. $\square$

4.3.6. *Exceptional Cases.* At this point there are only 6 remaining rules whose word-independence needs to be established and they come in pairs: 28 and 29, 32 and 40, and 152 and 184. These final 6 rules exhibit more intricate dynamics and the proofs are, of necessity, more delicate. We treat them in order of difficulty.

Proposition 4.36. *Rules 32 and 40 are word-independent.*

Proof. Let k be 32 or 40, let $\pi = (\pi_1, \pi_2, \ldots, \pi_n) \in S_n$ be a simple update order, and let $F\colon \mathbb{F}_2^n \to \mathbb{F}_2^n$ denote the SDS map $[\mathfrak{Wolf}_n^{(k)}, \pi]\colon \mathbb{F}_2^n \to \mathbb{F}_2^n$. The listed rules share the leftmost form shown in Figure 4.5 and it is easy to see that $\mathbf{0}$ is the only fixed state ($\mathbf{1}$ is not fixed and $a_2 = a_6 = 0$ means the rightmost 1 in any 1-block converts to 0 when updated). We also claim $\mathbf{0}$ is the only periodic state of F. Once this is established, the word-independence of $\mathfrak{Wolf}_n^{(k)}$ follows immediately from Proposition 4.6.

The values $a_0 = a_1 = a_4 = 0$ mean non-isolated 0-blocks persist indefinitely, they do not shrink or split. Moreover, $a_2 = a_6 = 0$ means that each non-isolated 0-block adds at least one 0 on its left-hand side with each application of F. In particular, any state $\mathbf{y} \neq \mathbf{0}$ with a non-isolated 0-block eventually becomes the fixed point $\mathbf{0}$. Thus no such $\mathbf{y}$ is periodic.

The rest of the argument is by contradiction. Suppose that $\mathbf{y}$ is a periodic point of F other than $\mathbf{0}$ and consider the i^{th} coordinates in $\mathbf{y}$, $F(\mathbf{y})$ and $F(F(\mathbf{y}))$. We claim that at least one of these coordinates is 0 and at least one of these is 1. This is because at least 4 out of the 5 local state configurations that do not involve non-isolated 0s change the coordinate (and when $k = 32$ all 5 of them make a change). The only way that y_i does not change value in $F(\mathbf{y})$ is if immediately prior to the application of $\mathsf{Wolf}_i^{(k)}$, the local state configuration is 011. Between this application of $\mathsf{Wolf}_i^{(k)}$ and the next, the 0 to the left is updated. It either is no longer isolated at this point (contradicting the periodicity of $\mathbf{y}$) or it now becomes a 1. In the latter case, the application of $\mathsf{Wolf}_i^{(k)}$ during the second iteration of F changes the i^{th} coordinate from 1 to 0. Note that we used the simplicity of the update order to ensure that each coordinate is updated only once

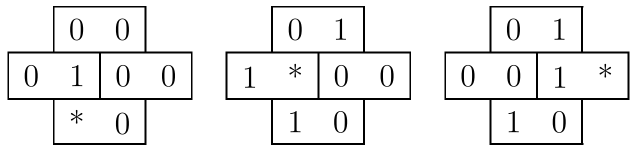


FIGURE 4.5. Three final pairs of word-independent Wolfram rules.

during each pass through F. Finally, suppose that $i = \pi_1$ and choose $\mathbf{y}$, $F(\mathbf{y})$ or $F(F(\mathbf{y}))$ so the $(i+1)^{\text{st}}$ coordinate is a 0. As soon as $\mathsf{Wolf}_{\pi_1}^{(k)}$ is applied, there is a non-isolated 0-block, contradicting the claim that $\mathbf{y} \neq \mathbf{0}$ is a periodic point. $\square$

We will now show that these four remaining rules agree with Wolfram rule 156 on their corresponding sets of periodic points, and then apply Proposition 4.2 to complete the proof of Theorem 4.15. We begin with a careful analysis of the evolution of the blocks of rule 156.

Example 4.37 (Wolfram rule 156)**.** Because the symmetric part of rule 156 is `--` no isolated blocks are ever created or destroyed and thus the number of blocks is invariant under iteration. Moreover, the four values $a_1 = a_5 = 0$ and $a_2 = a_3 = 1$ mean that substrings of the form 01 are fixed indefinitely, leaving the right end of every 0-block and the left end of every 1-block permanently unchanged. The other type of boundary can and does move since $p_3 = \mathtt{x}$, and it is its behavior that we want to examine. Let $\pi \in S_n$ be a simple update order and let $F \colon \mathbb{F}_2^n \to \mathbb{F}_2^n$ denote the SDS map $[\mathfrak{Wolf}_n^{(156)}, \pi] \colon \mathbb{F}_2^n \to \mathbb{F}_2^n$. So long as $\mathbf{y}$ is not $\mathbf{0}$ or $\mathbf{1}$, there is a 1-block followed by a 0-block and a corresponding substring of the form $01 \cdots 10 \cdots 01$. (If $\mathbf{y}$ only contains one 0-block and one 1-block, then the first two digits are the same as the last two digits, but that is irrelevant here.) As remarked above, the beginning of the 1-block and the end of the 0-block are fixed, but the boundary between them can vary.

Suppose both blocks are non-isolated and consider the central substring 10 at positions i and $i+1$. These are the only positions in the entire substring that can vary and the first one to be updated *will* change value. Assume the 0 is updated first. The 1-block grows, the 0-block shrinks and the boundary shifts one step to the right. As we cycle through the local functions, the simplicity of π guarantees that the $(i+2)^{\text{nd}}$ coordinate is updated before the $(i+1)^{\text{st}}$ coordinate is updated a second time. Thus the boundary shifts one more step to the right. This argument continues to be applicable until the 0-block shrinks to an isolated 0. At this point, the 0 is still updated before the 1 to its left is updated again, but this time the 0 remains unchanged. When the 1 to its left is updated it changes back to a 0, the 1-block shrinks, the 0-block grows and the boundary shifts to the left. The same argument with left and right reversed shows that now the 0-block continues to grow until the 1-block shrinks to an isolated 1, at which point the shifting stops and the boundary starts shifting back in the other direction.

Proposition 4.38. *Rules 152 and 184 are word-independent.*

Proof. Let k be 152 or 184, let $\pi = (\pi_1, \pi_2, \ldots, \pi_n) \in S_n$ be a simple update order, and let $F\colon \mathbb{F}_2^n \to \mathbb{F}_2^n$ denote the SDS map $[\mathfrak{Wolf}_n^{(k)}, \pi]\colon \mathbb{F}_2^n \to \mathbb{F}_2^n$. The listed rules share the second form shown in Figure 4.5 and it is easy to see that $\mathbf{0}$ and $\mathbf{1}$ are the only fixed states (since $a_2 = a_6 = 0$ means the rightmost 1 in any 1-block converts to 0 when updated). We also claim

$\mathbf{0}$ and $\mathbf{1}$ are the only periodic states of F. Once this is established, the word-independence of $\mathfrak{Wolf}_n^{(k)}$ follows immediately from Proposition 4.6.

Since isolated blocks are never created, the map ρ that sends $\mathbf{y}$ to the number of blocks it contains is a non-increasing potential function for $\mathfrak{Wolf}_n^{(k)}$. Moreover, since the only differences between rule 156 and rules 152 and 184 are that rule 152 removes isolated 1-blocks and rule 184 removes both isolated 1-blocks and isolated 0-blocks, the map F agrees with $[\mathfrak{Wolf}_n^{(156)}, \pi]$ so long as it is not called upon to update an isolated 1-block (or an isolated 0-block when $k = 184$). The long-term behavior of rule 156, however, as described in Example 4.37, shows that under iteration every $\mathbf{y}$ not equal to $\mathbf{0}$ or $\mathbf{1}$ eventually updates such an isolated block, removing it and decreasing ρ, thus showing that such a $\mathbf{y}$ is not periodic. $\square$

Finally, the argument for Wolfram rules 28 and 29 is a combination of the difficulties found in the proofs of Propositions 4.35 and 4.38.

Proposition 4.39. *Rules 28 and 29 are word-independent.*

Proof. Let k be 28 or 29, let $\pi = (\pi_1, \pi_2, \ldots, \pi_n) \in S_n$ be a simple update order, and let $F\colon \mathbb{F}_2^n \to \mathbb{F}_2^n$ denote the SDS map $[\mathfrak{Wolf}_n^{(k)}, \pi]\colon \mathbb{F}_2^n \to \mathbb{F}_2^n$. The listed rules share the rightmost form shown in Figure 4.5 and the values $a_5 = 0$ and $a_2 = 1$ mean that isolated blocks are never removed. Thus the map ρ that sends $\mathbf{y}$ to the number of blocks it contains is a non-decreasing potential function for $\mathfrak{Wolf}_n^{(k)}$. The four values $a_1 = a_5 = 0$ and $a_2 = a_3 = 1$ mean that substrings of the form 01 persist indefinitely, as in Wolfram rule 156. In fact, so long as ρ is unchanged, the behavior of F under iteration is indistinguishable from iterations of the map $[\mathfrak{Wolf}_n^{(156)}, \pi]$. Consider a substring of the form $01\cdots10\cdots01$ and suppose that the length of the 1-block on the left plus the length of the 0-block on the right is at least 4. We claim that any $\mathbf{y}$ containing such a substring is not periodic under F. If it were, the evolution of this substring would oscillate as described in Example 4.37 and at the point where the 0-block shrinks to an isolated 0, the 1-block on the left contains the substring 111. Moreover, between the point when that penultimate 0 becomes a 1 and the point when it is to switch back, the substring 111 is updated, increasing ρ. When k is 29, a similar increase in ρ can occur when the 1-block shrinks to an isolated 1 and the 0-block contains the substring 000. In neither case can a state containing a 1-block followed by a 0-block with combined length at least 4 be periodic under F.

Next, note that when $k = 29$ both of the constant states $\mathbf{0}$ and $\mathbf{1}$ are not fixed, but that for $k = 28$ $\mathbf{1}$ is not fixed, while $\mathbf{0}$ is fixed. Let P be the set of states containing both 0s and 1s that do not contain a 1-block followed by a 0-block with combined length at least 4, and, when $k = 28$, include the constant state $\mathbf{0}$ as well. Because we understand the way that such states $\mathbf{y} \in P$ evolve under Wolfram rule 156 (Example 4.37), we know that at no point in the future does a descendent of $\mathbf{y}$ ever contain a substring of the form 111 or 000. Thus P has been restricted enough to make the values of a_7 and a_0 irrelevant, and F sends P into itself. Thus for both $k = 28$ and 29, the local functions agree with those of the invertible rule $[\mathfrak{Wolf}_n^{(156)}, \pi]$ on P, and so by Proposition 4.2, rules 28 and 29 are word-independent. $\square$

# flips	0	1	2	3	4	5	6	7	8
# word-ind. rules	1	8	26	34	26	4	4	0	1
# rules	1	8	28	56	70	56	28	8	1
Percentage	100%	100%	93%	61%	37%	7%	14%	0%	100%

TABLE 3. The number of flips and the probability of word-independence.

4.3.7. *Flips and signature.* Now that the proof of Theorem 4.15 is complete, we pause to make a few comments about it and the 104 word-independent Wolfram rules. For each of the 8 local state configurations, Wolfram rule k either leaves the central coordinate unchanged or it "flips" its value. The number of local state configurations that are flipped in this way is strongly correlated with the probability that a given rule is word-independent, as shown in Table 3. The numbers in the third row are the binomial coefficients $\binom{8}{i}$, since they clearly count the number of Wolfram rules with exactly i flips. The key facts illustrated by Table 3 are that nearly all of the rules with at most 2 flips are word-independent, the percentage drops off rapidly between 2 and 6 flips, and word-independence is very rare among rules with 6 or more flips. In fact, all 5 such rules are word-independent because they are invertible. It would interesting to know whether this observation can be quantitatively (or even qualitatively) extended to a rigorous assertion about more general SDSs. In addition to counting the number of flips, one can keep track of the "sign" of the flips. More precisely, if we call a $0 \mapsto 1$ flip an *up-flip* and a $1 \mapsto 0$ a *down-flip*, then we can define the number of up-flips minus the number of down-flips to be the *signature* of a vertex function. This quantity can be normalized in several different ways: (i) dividing by the number of local state configurations gives a number that describes the average change in vertex state y_i as the vertex function is applied from a random $\mathbf{y} \in \mathbb{F}_2^n$, and *(ii)* dividing instead by the number of total flips describes the percentage of flips that increase (rather than decrease) the value of a state. If this second quantity is ± 1, then the local function is said to be a *one-way function.* These are precisely the functions such that the number of 0s (resp. 1s) is an SDS potential function, and thus all one-way functions are fixed point systems, and hence word-independent. Loosely speaking, it is intuitive to expect that the closer the quotient of the signature by the number of flips is to 0, the "more likely" there are non-fixed periodic points. Of course, there are obvious counterexamples, such as the ACA induced by the majority function (rule 232), which has 4 flips and signature 0, but always induces fixed point SDSs. A modified version of the signature, as described here, has been studied in [32]. Most of their results are experimental, and in the setting of classical cellular automata with parallel update. The emerging field of update order stochastic SDSs could be well-served by studying the signature.

4.3.8. *Periodic Points.* The proof of the classification of the 104 word-independent Wolfram rules, brings to light some interesting dynamical properties of those rules. In particular, we see how the Wolfram rules truly are local rules, in the sense that their set of periodic points have local characterizations, namely, from their 0- and 1-blocks. We shift our focus from the classification of these rules to understanding the periodic sets and the dynamics. We remark that we presented the proof of Theorem 4.15 as simply as possible, so in some cases, word-independence was proven non-constructively, in the sense that we did not explicitly give the set of periodic points. Determining these sets is not difficult, but at times tedious, and we omit the details. Most periodic point sets are easiest described by speaking of a set of all $\mathbf{y} \in \mathbb{F}_2^n$ that does *not* contain a collection of substrings, with the possible exception of the constant states $\mathbf{0}$ and $\mathbf{1}$. There are 28 subsets of $\mathbb{F}_2^n$ that are periodic points of some word-independent ACA. These sets, along with the list of rules whose periodic points are these sets, are listed in Table 8 in the appendix. The 28 sets in Table 8 are grouped by their distinct 15 equivalence classes. Moreover, some of these equivalence classes differ by only a constant state, $\mathbf{0}$ or $\mathbf{1}$.

Disregarding the constant states $\mathbf{0}$ and $\mathbf{1}$, there are nine non-empty proper subsets of $\mathbb{F}_2^n$ that appear in Table 8 up to equivalence. For each n, let $P_{n,i}$ denote the set of states in $\mathbb{F}_2^n$, defined by:

$P_{n,1}$: {No '11', '000'}

$P_{n,2}$: {No '11', '010'}

$P_{n,3}$: {No '11', '101'}

$P_{n,4}$: {No '000', '111', '1100'}

$P_{n,5}$: {No '000', '111'}.

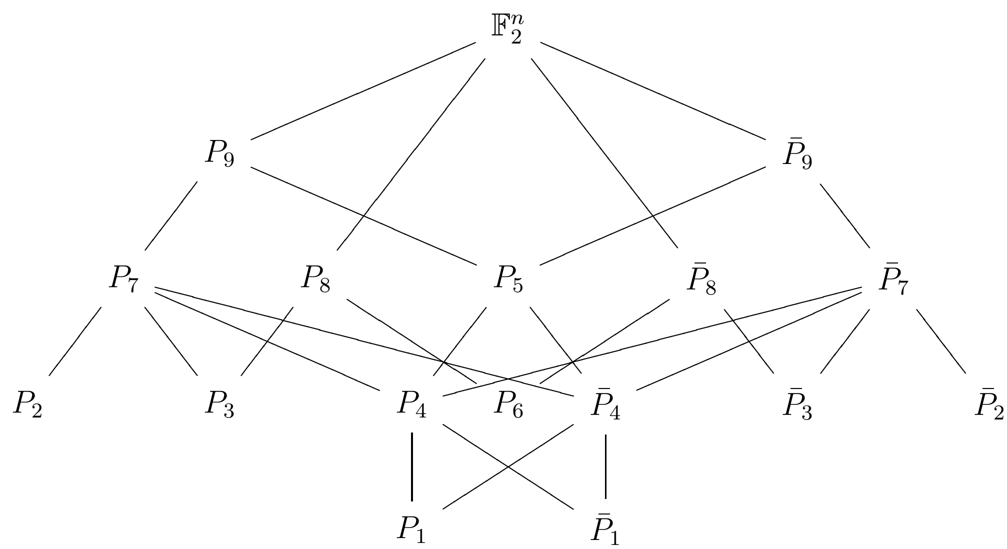
$P_{n,6}$: {No '101', '010'}

$P_{n,7}$: {No '11'}

$P_{n,8}$: {No '101'}

$P_{n,9}$: {No '111'}

For each $P_{n,i}$, let $\bar{P}_{n,i}$ denote its inversion, i.e., $\bar{P}_{n,i} = I(P_{n,i})$, where $I\colon \mathbb{F}_2^n \to \mathbb{F}_2^n$ is the inversion map from Definition 4.27. Notice that $\bar{P}_{n,i} = P_{n,i}$ only for $i \in \{5, 6\}$. These families of sets make up the *Wolfram poset*, under the relation of subset containment. We draw $\mathbb{F}_2^n$ and its covering relations, but omit $\emptyset$ (non-empty by including $\mathbf{0}$ or $\mathbf{1}$). Additionally, the subscript 'n' is omitted for clarity.



4.4. **Invariant sets.** Recall that proving word-independence of most of the 104 rules was straightforward. However, there were a few exceptional cases that we dealt with by determining the set of periodic points, finding an invertible rule that agreed with it on that set, and applying Theorem 4.8. It is interesting to ask whether this is a complete characterization of the word-independent rules, or in other words, whether the 104 rules are word-independent simply by virtue of agreeing with an invertible rule on an invariant set that contains their periodic points. This will also help us better understand how the rules are distributed on the Wolfram poset.

On a given set $P_{n,i}$, many of the 256 Wolfram rules agree with each other. For example, since there are no substrings of '11' or '000' in an element of $P_{n,1}$, then two Wolfram rules that differ only on the neighborhoods $\{111, 110, 011, 000\}$ will be the same when restricted to $P_{n,1}$. Thus there are only 16 distinct restrictions of Wolfram rules to $P_{n,1}$, and only one of these maps $P_{n,1}$ into itself, namely, the identity map. Consequently, if $\mathsf{Per}(\mathfrak{Wolf}_n^{(k)})$ is contained in $P_{n,1}$, then $\mathsf{Fix}(\mathfrak{Wolf}_n^{(k)}) = \mathsf{Per}(\mathfrak{Wolf}_n^{(k)}) = P_{n,1}$, and rule k is word-independent.

For every word-independent Wolfram rule k, let $E(k)$ be the set of rules that agree with rule k on $\mathsf{Per}(\mathfrak{Wolf}_n^{(k)})$. It is not difficult to compute $E(k)$ for every Wolfram rule. We have done this, and point out several interesting observations about these sets.

Remark 4.40. Every set $E(k)$ contains an invertible rule.

Remark 4.41. If the set of non-constant periodic points of rule k is $P_{n,i}$ (or $\bar{P}_{n,i}$) for $i \neq 3$, then $E(k)$ contains only word-independent rules.

Remark 4.40 suggests that the method used to prove the exceptional cases of Theorem 4.15 can be used to prove it for all Wolfram rules that are not automatically word-independent by either having $\mathsf{Per}(\mathfrak{F}_Y) = \mathbb{F}_2^n$ or only constant states. Specifically, there are three classes of periodic points:

- $\mathsf{Per}(\mathfrak{Wolf}_n^{(k)}) = \{\mathbf{0}\}$, $\{\mathbf{1}\}$, or $\{\mathbf{0}, \mathbf{1}\}$,
- $\mathsf{Per}(\mathfrak{Wolf}_n^{(k)}) = \mathbb{F}_2^n$,
- $\mathsf{Per}(\mathfrak{Wolf}_n^{(k)}) = P_{n,i}$ or $\bar{P}_{n,i}$ (plus possibly $\{\mathbf{0}\}$ or $\{\mathbf{1}\}$).

In the first two cases, $\mathfrak{Wolf}_n^{(k)}$ is trivially word-independent. In the third case, Remark 4.41 says that there exists an invertible rule that agrees with it on $\mathsf{Per}(\mathfrak{Wolf}_n^{(k)})$, and word-independence follows from Theorem 4.8.

Remark 4.41 is interesting because it shows that a much stronger result of Theorem 4.8 nearly holds for the special case of ACAs. By a special case of Theorem 4.8, in order to prove that a given rule k is word-independent, it suffices to find M and an invertible rule ℓ that agrees with rule k on M. However, for nearly every ACA, the much weaker condition that rule ℓ need simply be word-independent suffices.

4.5. **Dynamics groups of ACAs.**

4.5.1. *Coxeter matrices.* Lemma 4.13 greatly simplifies the study of the dynamics groups of ACAs, because Wolfram rules are invariant over C_n, a transitive subgroup of $\mathsf{Aut}(\mathsf{Circ}_n) \cong D_n$. From Corollary 4.12, the Coxeter matrix of a fixed point ACAs is the identity matrix. Moreover, 26 of the 41 word-independent ACAs up to equivalence are fixed point systems. The following theorem describes the Coxeter matrices of the remaining 15 cases.

Theorem 4.42. *Let* $\mathfrak{Wolf}_n^{(k)}$ *be word-independent and assume* $\mathsf{Fix}(\mathfrak{Wolf}_n^{(k)}) \neq \mathsf{Per}(\mathfrak{Wolf}_n^{(k)})$. *Then*

$$m_{ij} = \begin{cases} 1\ , & i = j \\ 2\ , & |i-j| \neq 1 \mod n \\ m_k\ , & |i-j| = 1 \mod n\ . \end{cases}$$

where $m_k \in \{2, 3, 4, 6, 12\}$.

Proof. $\mathfrak{Wolf}_n^{(k)}$ is invariant under the transitive subgroup $C_n \leq \mathsf{Aut}(\mathsf{Circ}_n)$. By Lemma 4.13, if $i \neq j$, then $m_{ij} \geq 2$. Since the only edges in Circ_n are $\{i, i+1\}$, then if $|i-j| > 1$ modulo n, the functions F_i and F_j commute, which means that $m_{ij} \leq 2$. Together, we conclude that if $|i-j| > 1$, then $m_{ij} = 2$. Finally, consider the case when $|i-j| = 1$. Because Coxeter matrices are symmetric, and the only edges are of the form $\{i, i+1\}$, the only values of $m_{i,j}$ that remain are $m_{i,i+1}$. Let $h \in \mathsf{Aut}(\mathsf{Circ}_n)$ defined by $h : k \mapsto k+1$. Observe that

$$h \circ (F_i \circ F_{i+1}) \circ h^{-1} = F_{i+1} \circ F_{i+2} \ .$$

We conclude that $m_i m_{i+1} = m_{i+1} m_{i+2}$ for all i. In the statement of Theorem 4.42, we call this value m_k. Now, it only remains to determine the possible values of m_k. Of the 41 equivalence classes of word-independent Wolfram rules, only 15 of them have non-fixed points. Checking these cases (details omitted) confirms that $m_k \in \{2, 3, 4, 6, 12\}$. □

By Theorem 4.42, the Coxeter matrix of a word-independent Wolfram rule is completely determined by the value $m_{i,i+1}$, which can be either 1, 2, 3, 4, 6, or 12. The 104 word-independent rules were arranged in a very neat fashion by blocks in Table 1. By suitably rearranging the rows and columns, we obtain a table in which the rules are not quite as organized, but the values of $m_{i,i+1}$ are better grouped. This is done in Table 4. Notice that if $m_k \in \{3, 4, 12\}$, then $\mathfrak{Wolf}_n^{(k)}$ is invertible. Table 4 shows that most of the word-independent Wolfram rules are fixed point systems, and thus have a trivial dynamics group. We now analyze the remaining rules.

4.5.2. *Abelian and linear dynamics groups.* There are a few Wolfram rules for which the dynamics groups are readily computed to be well-known groups.

Proposition 4.43. *For $k \in \{28, 29, 51\}$ and $n > 4$, $G(\mathfrak{Wolf}_n^{(k)}) \cong C_2^n$ and $H(\mathfrak{Wolf}_n^{(k)}) \cong C_2$.*

	p_3	x	-	x	-	-	0	0	-	1	1
	p_2	x	x	-	-	0	-	0	1	-	1
p_4p_1		18	66	24	72	64	8	0	74	88	90
0x	1				6	6	6	6			
x1	37				6				6	6	6
-x	129	12	4	4	6	6	6	6			
x-	36	12	4	4	6				6	6	6
xx	33	2	12	12	3						
--	132	3	1	1	1	1	1	1	1	1	1
-0	128		1	1	1	1	1	1	1	1	1
1-	164		1	1	1	1	1	1	1	1	1
10	160		1	1	1	1	1	1	1	1	1
-1	133		2	2	1	1	1	1	1	1	1
0-	4		2	2	1	1	1	1	1	1	1
01	5		2	2	1	1	1	1	1	1	1
00	0				1	1	1	1			
11	165				1				1	1	1
1x	161								1	1	1
x0	32					1	1	1			

TABLE 4. The 104 Wolfram rules, and the value $m_{i,i+1}$.

Proof. The only fixed points of these rules are the alternating states, i.e., $0101\cdots01$ and $1010\cdots10$. Thus the number of non-fixed periodic points is equal to $|P_{n,3}|$ when n is odd, and $|P_{n,3}|-2$ when n is even. This number is zero for $n=4$ (leading to a trivial dynamics group), and is at least as large as n for $n>4$. For rules 28 and 29, $m_{i,i+1}=2$, and so the dynamics group is abelian, with every generator having order 2. Thus the map defined by

$$\iota : G(\mathfrak{Wolf}_n^{(k)}) \longrightarrow C_2^n, \qquad \iota : F_i \longmapsto e_i$$

is an isomorphism. Also, for any simple update order $\pi \in S_n$, ι maps $[\mathfrak{Wolf}_n^{(k)}\pi]$ to the diagonal element in C_2^n, and so $H(\mathfrak{Wolf}_n^{(k)}) \cong C_2$. We remark that rule 51 is the "local inversion map" $\overline{\mathsf{id}}$ from Example 4.10, where we verified that its full and permutation dynamics groups are C_2^n and C_2, respectively. $\square$

It is perhaps surprising that rules 28 and 29 were among the six exceptional cases that we dealt with separately in the proof of Theorem 4.15 due to complications in analyzing their dynamics, yet they have very simple dynamics groups.

Proposition 4.44. $G(\mathfrak{Wolf}_n^{(102)}) \cong \mathrm{GL}(n, 2)$.

Proof. Rule 102 is linear, with $\mathsf{wolf}_n^{(102)} : (y_{i-1}, y_i, y_{i+1}) = y_i + y_{i+1}$. The matrix representation of the update rule at vertex i is $I_n + E_{i,i+1}$, where I_n is the $n \times n$ identity matrix, and $E_{i,i+1}$ is the matrix everywhere 0 except for 1 in the $(i, i+1)$ entry. It is known [41, pg. 455] that the n matrices of this form generate $\mathrm{GL}(n, 2)$, and thus $G(\mathfrak{Wolf}_n^{(102)}) \cong \mathrm{GL}(n, 2)$. $\square$

We have used a computer program to compute $H(\mathfrak{Wolf}_n^{(102)})$ for small values of n, and for $4 \leq n \leq 8$, it is $\mathrm{GL}(n, 2)$ as well. This likely can be proven by showing that a certain set of matrices generates $\mathrm{GL}(n, 2)$, but we have not yet been able to show this.

4.5.3. *Alternating and symmetric dynamics groups.* There are a handful of Wolfram rules whose dynamics groups we have computed for $4 \leq n \leq 8$, and are either the alternating group or the symmetric group. The size of these groups depend on the number of non-fixed periodic points. In this section, we state our conjectures, which are based on the assumption that these patterns hold for larger n.

Conjecture 4.45. $G(\mathfrak{Wolf}_n^{(57)}) \cong H(\mathfrak{Wolf}_n^{(57)}) \cong A_n$ and $G(\mathfrak{Wolf}_n^{(54)}) \cong H(\mathfrak{Wolf}_n^{(54)}) \cong A_{n-1}$.

Rules 54 and 57 are invertible. Rule 54 has no fixed points, and the only fixed point of rule 57 is the constant state $\mathbf{0}$.

The next class of rules are characterized by having non-fixed periodic point set $P_{n,7}$ (or $\bar{P}_{n,7}$), which is the set of n-bit cyclic strings that have non-isolated 1s (or 0s). It is well-known [19] that the cardinality of this set is counted by the Lucas numbers $L(n)$. The Lucas numbers, like the Fibonacci numbers $F(n)$, are defined by the recurrence $a_n = a_{n-1} + a_{n-2}$, with the difference being the initial values. The first two Fibonacci numbers are defined to be $F(0) = 0$ and $F(1) = 1$, whereas the first two Lucas numbers are $L(0) = 2$ and $L(1) = 2$. These observations about $P_{n,7}$ tell us that the dynamics groups of these rules must be contained in the symmetric group $S_{L(n)}$.

Conjecture 4.46. If $k \in \{1, 9, 110, 126\}$, then

$$G(\mathfrak{Wolf}_n^{(k)}) \cong \begin{cases} A_{L(n)} & F(n-1) \text{ even} \\ S_{L(n)} & F(n-1) \text{ odd,} \end{cases}$$

where $L(n)$ is the n^{th} Lucas number, and $F(n)$ is the n^{th} Fibonacci number.

One perhaps interesting part of this conjecture is how the parity of the Fibonacci numbers determines whether the group is the alternating or symmetric group. Consider rule 1. Each local update function $\mathsf{Wolf}_i^{(1)}$, being an involution, is a product of disjoint transpositions. To count how many, notice that for $\mathbf{y} \in P_{n,7}$, $\mathsf{Wolf}_i^{(1)}(\mathbf{y}) \neq \mathbf{y}$ iff $y_{i-1} = y_{i+1} = 0$. The number of disjoint transpositions of $\mathsf{Wolf}_i^{(1)}$ is the number of pairs of states of this form that there are in P_7. This number is precisely the number of substrings of the remaining $n-3$ vertices that do not contain consecutive 1s. When $n=4$, this is 2, the 3rd Fibonacci number, and when $n=5$, this is 3, the 4th Fibonacci number. Also, it is easy to see that this number satisfies the recurrence relation $a_n = a_{n-1} + a_{n-2}$. Therefore, $\mathsf{Wolf}_i^{(1)}$ is a product of an even number of disjoint transpositions iff $F(n-1)$ is even. This shows that $G(\mathfrak{Wolf}_n^{(1)}) \leq A_{L(n)}$ when $F(n-1)$ is even.

It remains to show that when $F(n-1)$ is odd, we can obtain any transposition, and when $F(n-1)$ is even, we can obtain any 3-cycle. Also, we must argue why this holds for rules 9, 110, and 126 as well.

There are five Wolfram rules up to equivalence that we have not yet dealt with: rules 150, 105, 78, 108, and 156. Four of these five come in pairs, which we will discuss separately.

4.5.4. *Exceptional cases: Rules 150 and 105.* Rules 150 and 105 are both invertible, and are closely related. Rule 150 is the parity function par_3 and rule 105 is the negation of parity, $\overline{\mathsf{par}}_3$ which are defined by

$$\mathsf{par}_3, \overline{\mathsf{par}}\colon \mathbb{F}_2^3 \longrightarrow \mathbb{F}_2, \qquad \mathsf{par}_3(x) = \sum_{i=1}^{3} x_i \ , \qquad \overline{\mathsf{par}}_3(x) = 1 + \sum_{i=1}^{3} x_i \ .$$

These are also the two rules with $m_{i,i+1} = 3$. In light of this, it is not surprising that the dynamics groups of these two rules are related as well. We have not pinned down these groups, but have computed their orders for small values of n, and have conjectured their orders for all n. With the help of a computer program, we have computed the size of both the full dynamics group G, and the permutation dynamics group H for $4 \leq n \leq 8$. We report in Table 5 the value of $|G|$, and for convenience, the index $[G : H]$. The

$k = 150$	$\|G\|$	$[G : H]$
$n = 4$	$4! \cdot 2^{4-2}$	2
$n = 5$	$5! \cdot 2^{4-1}$	1
$n = 6$	$6! \cdot 2^{4-2}$	2
$n = 7$	$7! \cdot 2^{4-1}$	1
$n = 8$	$8! \cdot 2^{4-2}$	

$k = 105$	$\|G\|$	$[G : H]$
$n = 4$	$4! \cdot 2^{4-2}$	2
$n = 5$	$5! \cdot 2^{4-1}$	1
$n = 6$	$6! \cdot 2^{4-1}$	2
$n = 7$	$7! \cdot 2^{4-1}$	1
$n = 8$	$8! \cdot 2^{4-2}$	

Table 5. The orders of the dynamics groups of rule 150 (parity) and rule 105 (1+parity).

slight differences in the order of the groups is due to the fact that for certain values of n, rules 150 and 105 have a different number of fixed points, which don't contribute to the dynamics group. The fixed points of rule 150 are the constant states $\mathbf{0}$ and $\mathbf{1}$, and if n is even, then additionally the alternating states $1010\cdots10$ and $0101\cdots01$. The fixed points of rule 105 are all the form $1100\cdots1100$ and if n is a multiple of 4, then there are 4 of these, otherwise there are none. This insight leads to the following conjecture.

Conjecture 4.47. The order of the dynamics groups of rules 150 and 105 are

$$|G(\mathfrak{Wolf}_n^{(105)})| = \begin{cases} n! \cdot 2^{n-2} & 4 \mid n \\ n!\, 2^{n-1} & 4 \nmid n \end{cases} \qquad |G(\mathfrak{Wolf}_n^{(150)})| = \begin{cases} n! \cdot 2^{n-2} & 2 \mid n \\ n!\, 2^{n-1} & 2 \nmid n. \end{cases}$$

It is also reasonable to conject that if n is even, then $[G : H] = 2$, and if n is odd, then $G = H$. If this is true, as the computational results suggest, it can likely be explained by showing that H is a product of n

odd permutations, and so H contains only the even permutations of G. We conclude this section by pointing out that the dynamics group of Rule 150 is isomorphic to the subgroup of $\mathrm{GL}(n,2)$ that is generated by matrices of the form $I + E_{i,i-1} + E_{i,i+1}$, where $E_{i,j}$ is the matrix of all 0s, except the (i,j)-entry is 1.

4.5.5. *Exceptional cases: Rules 73 and 108.* Another pair of rules that have complicated, but related dynamics groups are rules 73 and 108. The periodic point set of rule 73 is $P_{n,9}$, which is the largest periodic point set that is not $\mathbb{F}_2^n$. On this set, rule 73 agrees with rule 108, which is invertible. In light of this, it is reasonable to expect that the dynamics groups of these two rules to have some similarities. We used a computer program to calculate the order of the full dynamics group and the permutation dynamics group for $4 \leq n \leq 7$, which are shown in Table 6. We note that neither

$k=73$	$\lvert G\rvert$	$[G:H]$
$n=4$	$7!/2$	1
$n=5$	$16 \cdot 11!$	16
$n=6$	$11664 \cdot 18!$	32
$n=7$	$10^6 \cdot 2^{13} \cdot 3^6 \cdot 30!$	64

$k=108$	$\lvert G\rvert$	$[G:H]$
$n=4$	$7!/2$	1
$n=5$	$16 \cdot 11!$	16
$n=6$	$23328 \cdot 18!$	32
$n=7$	$10^6 \cdot 2^{13} \cdot 3^{13} \cdot 30!$	64

Table 6. The orders of the dynamics groups of rule 73 and rule 108.

of these sequences of $|G|$ for increasing n exists in Neal Sloane's Database of Integer Sequences [38], making it unlikely that the dynamics groups are comprised of a common class of groups.

4.5.6. *Exceptional cases: Rule 156.* We computed the order of the dynamics groups of rule 156 for $4 \leq n \leq 7$, which are shown in Table 7, and are significantly larger than those of any other rule. As with rules 78 and 108,

$k = 156$	$\lvert G \rvert$	$[G : H]$
$n = 4$	648	8
$n = 5$	7962624	16
$n = 6$	2176792336000000	32
$n = 7$	786040603451496383643648000000	64

TABLE 7. The orders of the dynamics groups of rule 156.

the sequence of the orders of G for increasing n does not currently appear in [38].

4.6. **Concluding remarks about ACAs.** To gain a clearer picture of the dynamics of ACAs, it is worthwhile to study the 152 non-word-independent rules as well. However, this is a large task is is relegated to a future research project. We summarize our analysis of the 104 word-independent Wolfram rules in Table 2 in the Appendix, by listing the decimal representation, tag, dynamically equivalent rules, number of flips, signature, Coxeter number, dynamics groups, and periodic point set, of each rule from a transversal. The rules are grouped into three blocks, with first block containing the the symmetric ones, the second block containing the quasi-symmetric rules, and the third containing the remaining rules. Though our study of determining the dynamics group is incomplete, we list the conjectured groups for those that we think we know. We comment in hindsight on a few properties of these rules that can be read off of Table 2. First of all, most of the rules (82 out of 104) are word-independent by virtue of by either being invertible, or a fixed point system. In fact, we can conclude the following.

Remark 4.48. If the set of non-constant (not **0** or **1**) periodic points of a word-independent set of Wolfram rules $\mathfrak{Wolf}_n^{(k)}$ is $P_{n,1}$, $P_{n,2}$, $P_{n,3}$, $P_{n,5}$, $P_{n,6}$, $P_{n,8}$, or an inversion of one of these, then $\mathsf{Fix}(\mathfrak{F}_Y) = \mathsf{Per}(\mathfrak{F}_Y)$.

Through this study of ACAs, we gained a better understanding of the 104 word-independent Wolfram rules and how they are related by their sets of periodic points, and a good number of the techniques and ideas developed in this section can be extended to general SDSs. We learned that roughly 40% of ACAs are word-independent, and it would be interesting to investigate whether this holds for systems over other families of graphs. The concepts such as flips and signature likely are useful in the study of stochastic SDSs. Thus this study is a good starting point for exploring in either of these directions because of the simplicity of ACAs.

5. Stochastic Sequential Dynamical Systems

Most of the SDS literature has assumed that the local functions, underlying graph, and update order, are deterministic. In this section we lay the groundwork for the study of stochastic sequential dynamical systems, or "StSDSs." There are many ways to add stochasticity to an SDS. The dependency graph, functions, or update order can all be chosen from a distribution. Here, we will only consider the case where the dependency graph is chosen from a random graph model, and the update order is chosen from a distribution. There are also many different properties and behaviors about stochastic SDSs that can be studied. In this section, we will look at how symmetries in the update order distribution are reflected in symmetries in the phase space.

5.1. **Preliminaries.** Throughout this section, we will only be considering simple update orders. Though these results can be extended to word update orders without much trouble, it complicates the notation without adding any deeper insight. Let $\mathcal{P} = \{(\pi_1, p_1), \ldots, (\pi_k, p_k)\}$ denote a probability distribution of update orders, i.e., update order π_i has associated probability p_i. For ease of notation, we will omit a (π_i, p_i) pair from $\mathcal{P}$ if $p_i = 0$, and so $1 \leq |\mathcal{P}| \leq n!$.

Definition 5.1 (Balanced random graph model)**.** Let $\mathcal{Y}$ be a graph on n vertices. A random graph model $G(\mathcal{Y})$ is said to be *balanced* if isomorphic subgraphs have the same probability.

The key property of balanced random graph models that we will use is that if $\sigma \in \mathsf{Aut}(\mathcal{Y})$, then $G(\mathcal{Y})$ and $G(\sigma(\mathcal{Y}))$ are identical models. Two common balanced random graph distributions are $G_{n,p} = G_{n,p}(K_n)$ and $G_{n,M} = G_{n,M}(K_n)$. In $G_{n,p}(\mathcal{Y})$, an element $Y \in G_{n,p}(\mathcal{Y})$ has the same vertex set as $\mathcal{Y}$, and the probability that Y contains the edge $\{i,j\}$ is p if $\{i,j\} \in e[\mathcal{Y}]$, and 0 otherwise. Thus, $Y \in G_{n,p}(\mathcal{Y})$ (where $Y \leq \mathcal{Y}$) has probability

$$\Pr(Y) = p^{|e[Y]|} \cdot (1-p)^{|e[\mathcal{Y}]|-|e[Y]|}.$$

In $G_{n,M}(\mathcal{Y})$, every subgraph with exactly M edges has the same probability, so $Y \in G_{n,p}(\mathcal{Y})$ (with m edges) has probability

$$\Pr(Y) = \binom{|e[Y]|}{m}^{-1}.$$

Because we will be choosing the base graph from a distribution, the degree of a vertex depends on which random graph is chosen. Thus it is reasonable to impose the blanket hypothesis that all sequences of local functions are symmetric (they depend only the multiset of vertex states) and homogeneous (if $f_i, f_j \colon K^d \to K$, then $f_i = f_j$).

Definition 5.2 (Stochastic SDS)**.** A *stochastic sequential dynamical system* (StSDS) is a triple $(G(\mathcal{Y}), \mathfrak{F}_Y, \mathcal{P})$ consisting of a balanced random graph model $G(\mathcal{Y})$, a set of homogeneous symmetric local functions $\mathfrak{F}_Y$, and a distribution $\mathcal{P}$ of update orders.

A classical SDS has an associated SDS map $K^n \to K^n$, and this is encoded by the phase space, $\Gamma[\mathfrak{F}_Y, \pi]$. Since the phase space is a directed graph where the out-degree of every vertex is 1, its adjacency matrix may be viewed as a very simple Markov chain, which we will denote by $\mathsf{M}[\mathfrak{F}_Y, \pi]$. We bring this up because it is more convenient to speak of a stochastic SDS map as a Markov chain than as a stochastic map $K^n \to K^n$, or an edge-weighted digraph. The Markov chain of a stochastic SDS is the weighted average of the Markov chains of the individual SDSs, i.e.,

$$\mathsf{M}[\mathfrak{F}_{G(\mathcal{Y})}, \mathcal{P}] = \sum_{Y_i \in G(\mathcal{Y})} \sum_{(\pi_j, p_j) \in \mathcal{P}} p_j \Pr(Y_i) \cdot \mathsf{M}[\mathfrak{F}_{Y_i}, \pi_j] = \sum_{i,j} C_{ij} \cdot \mathsf{M}[\mathfrak{F}_{Y_i}, \pi_j] \,,$$

where we define $C_{ij} = p_j \cdot \Pr(Y_i)$. The Markov chain is a $|K^n| \times |K^n|$ matrix, and we shall denote its entries by $m(\mathbf{y}, \mathbf{z})$, where $\mathbf{y}, \mathbf{z} \in K^n$.

We will point out several special cases of stochastic SDS maps. An StSDS map over a fixed base graph Y is the special case $[\mathfrak{F}_{G(Y)}, \mathcal{P}]$ when $G(Y) = G_{n,p}(Y)$ and $p = 1$, or alternatively, when $G(Y) = G_{n,M}(Y)$ and $M = |e[Y]|$. We will abbreviate the associated map as simply $[\mathfrak{F}_Y, \mathcal{P}]$. Also, an StSDS with a fixed update order π is the special case when $\mathcal{P} = \{(\pi, 1)\}$, which we will abbreviate as $[\mathfrak{F}_{G(\mathcal{Y})}, \pi]$. Thus, the classical non-stochastic SDS map is

$$[\mathfrak{F}_Y, \pi] = [\mathfrak{F}_{G(Y)}, \{(\pi, 1)\}]$$

where $G(Y) = G_{n,p}(Y)$ and $p = 1$.

5.2. **Dynamical equivalence of StSDSs.** Recall that two SDSs are dynamically equivalent if their maps are related by conjugation of a bijection $K^n \to K^n$, which is equivalent to the two Markov chains being similar matrices. We pause to restate the result of Proposition 2.9 in terms of Markov chains. The result of Proposition 2.9 was that if $\varphi \in \mathsf{Aut}(Y)$, then

$$[\mathfrak{F}_Y, \varphi * \pi] = \varphi \circ [\mathfrak{F}_Y, \pi] \circ \varphi^{-1} ,$$

where an element $\varphi \in \mathsf{Aut}(Y)$ composed with a map $K^n \to K^n$ was defined by the action from (1.4). In the language of Markov chains, this means that

$$\mathsf{M}[\mathfrak{F}_Y, \varphi * \pi] = \mathsf{P}_\varphi \circ \mathsf{M}[\mathfrak{F}_Y, \pi] \circ \mathsf{P}_\varphi^{-1} , \tag{5.1}$$

where P_φ is an associated permutation matrix, which we will describe explicitly. The idea is that each row and column corresponds with a point in K^n, and the matrix P_φ permutes the rows and columns according to the permutation of φ on the coordinates of K^n. We now define this formally.

Definition 5.3 (Induced permutation)**.** For $\varphi \in S_n$, the *induced permutation* of φ, denoted $\tilde{\varphi} \in S_{K^n}$, is the permutation of K^n that arises from the extension of the action of φ on the standard basis vectors, i.e.,

$$\{\varphi \cdot e_1, \ldots, \varphi \cdot e_n\} = \{e_{\varphi^{-1}(1)}, \ldots, e_{\varphi^{-1}(n)}\} ,$$

to all of K^n. For a fixed ordering of K^n, let P_φ be the permutation matrix of $\tilde{\varphi}$. If conjugation by P_φ preserves a Markov chain M, then we say that the induced permutation $\tilde{\varphi}$ preserves M.

As an example suppose $\sigma = (1\,2) \in S^4$, and $K = \mathbb{F}_2$. Then $\tilde{\sigma} \in S_{K^4}$ is the permutation that swaps the first two entries of elements in K^4, i.e.,

$$\tilde{\sigma} = (0100\ \ 1000)\,(0101\ \ 1001)\,(0110\ \ 1010)\,(0111\ \ 1011) \in S_{K^4}\ .$$

Since any permutation is a composition of transpositions, and any permutation of the vertices of the phase space can be viewed as conjugation of the Markov chain by a permutation matrix, we may view any induced permutation as a sequence of steps where we swap the i^{th} row of the Markov chain with j^{th} row, and the i^{th} column of the Markov chain with j^{th} column. This operation is easily seen to preserve the matrix, except for the 2×2-minor corresponding to the i^{th} and j^{th} entries, which get permuted as follows:

$$\begin{bmatrix} m_{ii} & m_{ij} \\ m_{ji} & m_{jj} \end{bmatrix} \longmapsto \begin{bmatrix} m_{jj} & m_{ji} \\ m_{ij} & m_{ii} \end{bmatrix}\ . \tag{5.2}$$

Remark 5.4. Let $(i\ j) \in S_n$, and let the induced permutation in S_{K^n} be

$$(i_1\ j_1)(i_2\ j_2)\cdots(i_m\ j_m)\ .$$

Then the StSDS map is fixed under the action of the transposition $(i\ j)$ if and only if for every $k = 1, \ldots, m$,

$$m_{i_k,l} = m_{j_k,l} \quad \text{and} \quad m_{l,i_k} = m_{l,j_k}\ , \qquad \text{for all } l = 1, \ldots, |\mathbb{F}^n|\ .$$

This is equivalent to saying that the i_k^{th} and j_k^{th} rows and columns are identical, except for the 2×2 $i_k j_k$-minor, which must be symmetric and with the same diagonal entries. To check whether an element $\sigma \in S_n$ is an StSDS automorphism, first break it into transpositions, and then apply the above steps successively, and see if the resulting operations preserve the Markov chain.

This can naturally be extended to stochastic SDSs.

Definition 5.5 (Dynamical equivalence of StSDSs)**.** Two stochastic SDSs $(G(\mathcal{Y}), \mathfrak{F}_{G(\mathcal{Y})}, \mathcal{P})$ and $(G'(\mathcal{Y}), \mathfrak{F}'_{G(\mathcal{Y})}, \mathcal{P}')$ are *dynamically equivalent* if there is an invertible $|K^n| \times |K^n|$ matrix P such that

$$\mathsf{M}[\mathfrak{F}_{G(\mathcal{Y})}, \mathcal{P}] = \mathsf{P} \cdot \mathsf{M}[\mathfrak{F}'_{G'(\mathcal{Y})}, \mathcal{P}'] \cdot \mathsf{P}^{-1} .$$

The action of S_n on the set of update orders from (1.4) can be extended to a distribution of update orders, by

$$\varphi * \mathcal{P} = \{(\varphi * \pi_1, p_1), \ldots, (\varphi * \pi_k, p_k)\} .$$

Now, Proposition 2.9 can extended rather easily to stochastic SDSs.

Proposition 5.6. *Let $\mathcal{Y}$ be a graph, and $\varphi \in \mathsf{Aut}(\mathcal{Y})$. For any update order distribution $\mathcal{P}$, the StSDS $(G(\mathcal{Y}), \mathfrak{F}_{G(\mathcal{Y})}, \mathcal{P})$ is dynamically equivalent to $(G(\mathcal{Y}), \mathfrak{F}_{G(\mathcal{Y})}, \varphi * \mathcal{P})$, and the equivalence is given by*

$$\mathsf{M}[\mathfrak{F}_{G(\mathcal{Y})}, \varphi * \mathcal{P}] = \mathsf{P}_\varphi \cdot \mathsf{M}[\mathfrak{F}_{G(\mathcal{Y})}, \mathcal{P}] \cdot \mathsf{P}_\varphi^{-1}. \tag{5.3}$$

Proof. The proof follows from breaking up the StSDS map into a linear combination of individual SDS maps, and applying Proposition 2.9, to get

$$\begin{aligned}
\mathsf{M}[\mathfrak{F}_{G(\mathcal{Y})}, \varphi * \mathcal{P}] &= \sum_{Y_i \in \mathcal{Y}} \sum_{(\pi_j, p_j) \in \mathcal{P}} C_{i,j} \mathsf{M}[\mathfrak{F}_{Y_i}, \varphi * \pi_j] \\
&= \sum_{Y_i \in \mathcal{Y}} \sum_{(\pi_j, p_j) \in \mathcal{P}} C_{i,j} \left(\mathsf{P}_\varphi \cdot \mathsf{M}[\mathfrak{F}_{Y_i}, \pi_j] \cdot \mathsf{P}_\varphi^{-1} \right) \\
&= \mathsf{P}_\varphi \cdot \left(\sum_{Y_i \in \mathcal{Y}} \sum_{(\pi_j, p_j) \in \mathcal{P}} C_{i,j} \mathsf{M}[\mathfrak{F}_{Y_i}, \pi_j] \right) \cdot \mathsf{P}_\varphi^{-1} \\
&= \mathsf{P}_\varphi \cdot \mathsf{M}[\mathfrak{F}_{G(\mathcal{Y})}, \mathcal{P}] \cdot \mathsf{P}_\varphi^{-1},
\end{aligned}$$

and hence the proposition is proven. $\square$

5.3. **Stabilizer subgroups.** Proposition 5.6 tells us that $\mathsf{Aut}(\mathcal{Y})$ acts on the set of StSDSs over $G(\mathcal{Y})$. For a classical SDS, if $\varphi \in \mathsf{Aut}(Y)$ is not the identity, then conjugation by the permutation matrix P_φ does not preserve the Markov chain. However, in the case of stochastic SDSs, it can happen that $\varphi * \mathcal{P} = \mathcal{P}$ for a non-identity φ, and thus

$$\mathsf{P}_\varphi \cdot \mathsf{M}[\mathfrak{F}_{G(\mathcal{Y})}, \mathcal{P}] \cdot \mathsf{P}_\varphi^{-1} = \mathsf{M}[\mathfrak{F}_{G(\mathcal{Y})}, \varphi * \mathcal{P}] = \mathsf{M}[\mathfrak{F}_{G(\mathcal{Y})}, \mathcal{P}] \; .$$

The subgroup of all such $\varphi \in \mathsf{Aut}(\mathcal{Y})$ is the stabilizer subgroup of this action. We will show some basic results about the symmetries of stochastic SDSs by analyzing their stabilizer subgroups. It is reasonable to expect that symmetries within the update order distribution will be reflected in symmetries in the dynamics, or the Markov chain, of the stochastic SDS. For example, consider a stochastic SDS over $G_{n,p}(K_n)$. If the update order is chosen uniformly from S_n, and K_n is fully symmetric, the n vertices are dynamically indistinguishable. It makes sense that $\varphi \in \mathsf{Aut}(K_n) \cong S_n$ acting on K^n should have no effect on the StSDS, or said differently, $\mathsf{Stab}[\mathfrak{F}_{G(\mathcal{Y})}, \mathcal{P}] \cong S_n$. In contrast, suppose that the update order distribution consists of a single ordering, π. Then in principle, we cannot necessarily expect the action of $\varphi \in S_n$ on K^n to preserve the dynamics, since we no longer have uniformity of the n vertices. In particular, vertex π_1 is updated first, a role played by none of the other vertices.

Definition 5.7. Let $H < S_n$, and fix $\pi \in S_{\mathcal{Y}}$. Define the update order distribution

$$\mathcal{P}_H^\pi = \left\{(h \cdot \pi, |H|^{-1}) \,:\, h \in H\right\} .$$

In many of the results, π is arbitrary, so we will denote such a set $\mathcal{P}_H^\pi$ simply by $\mathcal{P}_H$.

At first, it may seem unnatural and contrived to expect the update order distribution to have a group structure, but we do not find this unreasonable. For example, if the distribution is uniform over all of S_n, then there is a natural group structure, namely S_n. For a more complicated, and more convincing example, suppose that the underlying graph has n vertices arranged in a circle. If one were to pick the update order by first choosing a vertex at random, then updating in a clockwise fashion, the distribution would have a group structure of $C_n < S_n$. However, if one could update the vertices by traversing either clockwise or counterclockwise, then the distribution would have a group structure of $D_n < S_n$.

Proposition 5.8. *If $H < \mathsf{Aut}(\mathcal{Y})$, then*

$$H \leq \mathsf{Stab}[\mathfrak{F}_{G(\mathcal{Y})}, \mathcal{P}_H] \ .$$

Proof. Pick $\varphi \in H$. It follows easily that $\varphi * \mathcal{P}_H = \mathcal{P}_H$. By Proposition 5.6,

$$\mathsf{P}_\varphi \cdot [\mathfrak{F}_{G(\mathcal{Y})}, \mathcal{P}_H] \cdot \mathsf{P}_\varphi^{-1} = [\mathfrak{F}_{G(\mathcal{Y})}, \varphi * \mathcal{P}_H] = [\mathfrak{F}_{G(\mathcal{Y})}, \mathcal{P}_H] \ ,$$

and hence $\varphi \in \mathsf{Stab}[\mathfrak{F}_{G(\mathcal{Y})}, \mathcal{P}_H]$. □

In general, one should expect that $H \cong \mathsf{Stab}[\mathfrak{F}_{G(\mathcal{Y})}, \mathcal{P}_H]$. However, extra automorphisms can arise for certain simple or degenerate functions that will give rise to a larger stabilizer subgroup. For example, if $\mathfrak{F}_Y = \mathsf{Id}_Y$, the identity functions, then $\mathsf{Stab}[\mathfrak{F}_{G(\mathcal{Y})}, \mathcal{P}_H]$ will always be $\mathsf{Aut}(\mathcal{Y})$, regardless of H. Intuitively, it seems reasonable that as we increase the size of the group H, more symmetries should arise inside the StSDS, which is the motivation for the next theorem.

Proposition 5.9. *If $H < G < \mathsf{Aut}(\mathcal{Y})$, then*

$$\mathsf{Stab}[\mathfrak{F}_{G(\mathcal{Y})}, \mathcal{P}_H] < \mathsf{Stab}[\mathfrak{F}_{G(\mathcal{Y})}, \mathcal{P}_G] \ .$$

Proof. Let $\{g_i H\}$ be a complete set of coset representatives for H in G. (Note: H need not be normal in G). It easily follows that

$$\mathsf{M}[\mathfrak{F}_{G(\mathcal{Y})}, \mathcal{P}_G] = \frac{1}{[G:H]} \sum_{i=1}^{[G:H]} \mathsf{M}[\mathfrak{F}_{G(\mathcal{Y})}, g_i * \mathcal{P}_H].$$

Let $\varphi \in \mathsf{Stab}[\mathfrak{F}_{G(\mathcal{Y})}, \mathcal{P}_H]$. Since P_φ preserves $\mathsf{M}[\mathfrak{F}_{G(\mathcal{Y})}, \mathcal{P}_H]$ under conjugation, it suffices to show that P_φ preserves $\mathsf{M}[\mathfrak{F}_{G(\mathcal{Y})}, \mathcal{P}_G]$ under conjugation as well. By Proposition 5.6,

$$\begin{aligned}\mathsf{P}_\varphi \cdot \mathsf{M}[\mathfrak{F}_{G(\mathcal{Y})}, \mathcal{P}_G] \cdot \mathsf{P}_\varphi^{-1} &= \frac{1}{[G:H]} \sum_{i=1}^{[G:H]} \mathsf{P}_\varphi \cdot \mathsf{M}[\mathfrak{F}_{G(\mathcal{Y})}, g_i * \mathcal{P}_H] \cdot \mathsf{P}_\varphi^{-1} \\ &= \frac{1}{[G:H]} \sum_{i=1}^{[G:H]} \mathsf{M}[\mathfrak{F}_{G(\mathcal{Y})}, \varphi(g_i) * \mathcal{P}_H] \\ &= \mathsf{M}[\mathfrak{F}_{G(\mathcal{Y})}, \mathcal{P}_G]\ .\end{aligned}$$

The last equality holds because $\{\varphi(g_i)H\}$ is a complete set of coset representatives for H in G. We have thus shown that $\varphi \in \mathsf{Stab}[\mathfrak{F}_{G(\mathcal{Y})}, \mathcal{P}_G]$. □

5.4. **An example.** We conclude this section on stochastic SDSs by presenting an explicit example that illustrates the results of Propositions 5.8 and 5.9. Consider the random graph distribution $G_{3,2/3} := G_{n,p}(K_3)$ for $n = 3$ and $p = 2/3$, and let $\mathfrak{F}_Y = (\mathsf{Par}_k)$, the parity functions over $K = \mathbb{F}_2$. Though it is natural to have the vector v_k to correspond with the binary representation of k, we will deviate slightly, and swap the roles of v_3 and v_4, and use the following:

v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7
000	001	010	100	011	101	110	111

The motivation of this is to group the S_3-orbits together, so the symmetries of the Markov chain are more apparent. We also will show supplementary lines to emphasize the blocks of the Markov chain that exhibit such symmetries. Thus $S_{K_3} = S_8$, which we write as the group of permutations of the set $\{0, 1, 2, \ldots, 7\}$. The induced permutation of the transpositions (1 2), (2 3), and (4 5) are (1 2)(5 6), (2 3)(4 5), and (1 3)(4 6), respectively. Recall that P_φ for one of these transpositions in S_3 is the permutation matrix that permutes the rows and columns according the respective induced permutation.

Consider the 3-cycle $\varphi = (0\ 1\ 2)$, which generates $C_3 < S_3$, and let π be the update order $012 \in S_{K_3}$. Then

$$\mathcal{P}_{C_3} := \mathcal{P}_{C_3}^{\pi} = \{(012, 1/3), (120, 1/3), (201, 1/3)\} .$$

It is readily checked that $\mathsf{M}[\mathsf{Par}, G_{n,p}, 012] \neq \mathsf{M}[\mathsf{Par}, G_{n,p}, 120]$, but by Proposition 5.8, since $\varphi^i * \mathcal{P}_{C_3} = \mathcal{P}_{C_3}$ for any i, then the the stabilizer subgroup of the stochastic SDS is $\langle\varphi\rangle$. Equivalently,

$$\langle\varphi\rangle \hookrightarrow \mathsf{Stab}[\mathsf{Par}_{G_{n,p}(K_3)}, \mathcal{P}_{C_3}] \ , \tag{5.4}$$

and we can verify this explicitly by conjugating the Markov chain by P_φ:

$$[\mathsf{Par}_{G_{3,\frac{2}{3}}}, \mathcal{P}_{C_3}] = \frac{1}{3}[\mathsf{Par}_{G_{3,\frac{2}{3}}}, 012] + \frac{1}{3}[\mathsf{Par}_{G_{3,\frac{2}{3}}}, 120] + \frac{1}{3}[\mathsf{Par}_{G_{3,\frac{2}{3}}}, 201]$$

$$= \frac{1}{81}\begin{bmatrix} 27 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 6 & 10 & 0 & 0 & 4 & 4 \\ 0 & 0 & 3 & 6 & 6 & 12 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 6 & 10 & 8 \\ 0 & 10 & 0 & 4 & 5 & 2 & 6 & 0 \\ 0 & 6 & 12 & 0 & 0 & 5 & 2 & 2 \\ 0 & 0 & 6 & 0 & 12 & 0 & 5 & 4 \\ 0 & 8 & 0 & 4 & 4 & 2 & 0 & 9 \end{bmatrix} + \frac{1}{81}\begin{bmatrix} 27 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 6 & 0 & 0 & 6 & 12 & 0 \\ 0 & 0 & 3 & 0 & 10 & 0 & 6 & 8 \\ 0 & 6 & 10 & 3 & 4 & 0 & 0 & 4 \\ 0 & 6 & 0 & 0 & 5 & 12 & 0 & 4 \\ 0 & 0 & 4 & 10 & 6 & 5 & 2 & 0 \\ 0 & 12 & 0 & 6 & 2 & 0 & 5 & 2 \\ 0 & 0 & 4 & 8 & 0 & 4 & 2 & 9 \end{bmatrix}$$

$$+ \frac{1}{81}\begin{bmatrix} 27 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 6 & 10 & 0 & 8 \\ 0 & 10 & 3 & 6 & 0 & 4 & 0 & 4 \\ 0 & 6 & 0 & 3 & 12 & 0 & 6 & 0 \\ 0 & 0 & 6 & 12 & 5 & 2 & 0 & 2 \\ 0 & 0 & 0 & 6 & 0 & 5 & 12 & 4 \\ 0 & 4 & 10 & 0 & 2 & 6 & 5 & 0 \\ 0 & 4 & 8 & 0 & 2 & 0 & 4 & 9 \end{bmatrix} = \frac{1}{81}\left[\begin{array}{c|ccc|ccc|c} 81 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 9 & 12 & 10 & 6 & 16 & 16 & 12 \\ 0 & 10 & 9 & 12 & 16 & 16 & 6 & 12 \\ 0 & 12 & 10 & 9 & 16 & 6 & 16 & 12 \\ \hline 0 & 16 & 6 & 16 & 15 & 16 & 6 & 6 \\ 0 & 6 & 16 & 16 & 6 & 15 & 16 & 6 \\ 0 & 16 & 16 & 6 & 16 & 6 & 15 & 6 \\ \hline 0 & 12 & 12 & 12 & 6 & 6 & 6 & 27 \end{array}\right]$$

Proposition 5.8 tells us that for any element in $\langle\varphi\rangle$, the Markov chain is preserved under permutation of the rows and columns of according to the induced permutation $\tilde{\varphi}$. As previously mentioned, the induced permutations of the transpositions in S_3 are $(1\ 2)(5\ 6)$, and $(2\ 3)(4\ 5)$, and $(1\ 3)(4\ 6)$, and it is easily verified that neither of these induced permutations preserve the Markov chain (but the product of any two do), thus the stabilizer subgroup is precisely C_3.

To contrast this example, we will now consider a similar system, but using C_2 instead of C_3. In particular, for the transposition $\varphi = (1\ 2)$ and $\pi = 012$, we get the update order distribution $\mathcal{P}_{C_2} = \{(012, 1/2), (021, 1/2)\}$. The

Markov chain of this StSDS is

$$
\begin{aligned}
[\mathsf{Par}_{G_{3,\frac{2}{3}}}, \mathcal{P}_{C_2}] &= \frac{1}{2}[\mathsf{Par}_{G_{3,\frac{2}{3}}}, 012] + \frac{1}{2}[\mathsf{Par}_{G_{3,\frac{2}{3}}}, 021] \\
&= \frac{1}{54}\begin{bmatrix} 27 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 6 & 10 & 0 & 0 & 4 & 4 \\ 0 & 0 & 3 & 6 & 6 & 12 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 6 & 10 & 8 \\ 0 & 10 & 0 & 4 & 5 & 2 & 6 & 0 \\ 0 & 6 & 12 & 0 & 0 & 5 & 2 & 2 \\ 0 & 0 & 6 & 0 & 12 & 0 & 5 & 4 \\ 0 & 8 & 0 & 4 & 4 & 2 & 0 & 9 \end{bmatrix} + \frac{1}{54}\begin{bmatrix} 27 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 6 & 6 & 0 & 12 & 0 \\ 0 & 6 & 3 & 10 & 0 & 4 & 0 & 4 \\ 0 & 0 & 0 & 3 & 0 & 10 & 6 & 8 \\ 0 & 0 & 10 & 4 & 5 & 6 & 2 & 0 \\ 0 & 6 & 0 & 0 & 12 & 5 & 0 & 4 \\ 0 & 12 & 6 & 0 & 0 & 2 & 5 & 2 \\ 0 & 0 & 8 & 4 & 4 & 0 & 2 & 9 \end{bmatrix} \\
&= \frac{1}{54}\left[\begin{array}{c|ccc|ccc|c} 54 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 6 & 6 & 16 & 6 & 0 & 16 & 4 \\ 0 & 6 & 6 & 16 & 6 & 16 & 0 & 4 \\ 0 & 0 & 0 & 6 & 0 & 16 & 16 & 16 \\ \hline 0 & 10 & 10 & 4 & 10 & 8 & 8 & 0 \\ 0 & 12 & 12 & 0 & 12 & 10 & 2 & 6 \\ 0 & 12 & 12 & 0 & 12 & 2 & 10 & 6 \\ \hline 0 & 8 & 8 & 8 & 8 & 2 & 2 & 18 \end{array}\right]
\end{aligned}
$$

The induced permutation of $(1\ 2)$ is $(1\ 2)(5\ 6)$, and this preserves the Markov chain, as guaranteed by Proposition 5.8. However, the induced permutations of the other two transpositions do not, nor do their products. Together, this means that

$$C_2 \hookrightarrow \mathsf{Stab}[\mathsf{Par}, G_{3,\frac{2}{3}}, \mathcal{P}_{C_2}] \cong C_2\ .$$

Next, consider the update order distribution $\mathcal{P}'_{C_3} = \{(021, 1/3), (102, 1/3), (210, 1/3)\}$. Intuitively, $[\mathsf{Par}, G_{3,\frac{2}{3}}, \mathcal{P}'_{C_3}]$ should exhibit similar symmetries to that of $[\mathsf{Par}, G_{3,\frac{2}{3}}, \mathcal{P}_{C_3}]$. In fact, the Markov chains of these two StSDS are transposes of each other, and this makes sense, because the dynamics are equivalent, though are essentially done in reverse. Explicitly, the Markov chain is

$$[\mathsf{Par}, G_{3,\frac{2}{3}}, \mathcal{P}'_{C_3}] = \frac{1}{3}[\mathsf{Par}, G_{3,\frac{2}{3}}, 021] + \frac{1}{3}[\mathsf{Par}, G_{3,\frac{2}{3}}, 102] + \frac{1}{3}[\mathsf{Par}, G_{3,\frac{2}{3}}, 210]$$

$$= \frac{1}{81}\begin{bmatrix} 27 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 6 & 6 & 0 & 12 & 0 \\ 0 & 6 & 3 & 10 & 0 & 4 & 0 & 4 \\ 0 & 0 & 0 & 3 & 0 & 10 & 6 & 8 \\ 0 & 0 & 10 & 4 & 5 & 6 & 2 & 0 \\ 0 & 6 & 0 & 0 & 12 & 5 & 0 & 4 \\ 0 & 12 & 6 & 0 & 0 & 2 & 5 & 2 \\ 0 & 0 & 8 & 4 & 4 & 0 & 2 & 9 \end{bmatrix} + \frac{1}{81}\begin{bmatrix} 27 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 10 & 6 & 0 & 0 & 4 & 4 \\ 0 & 0 & 3 & 0 & 6 & 0 & 10 & 8 \\ 0 & 0 & 6 & 3 & 12 & 6 & 0 & 0 \\ 0 & 6 & 0 & 12 & 5 & 0 & 2 & 2 \\ 0 & 10 & 4 & 0 & 2 & 5 & 6 & 0 \\ 0 & 0 & 0 & 6 & 0 & 12 & 5 & 4 \\ 0 & 8 & 4 & 0 & 2 & 4 & 0 & 9 \end{bmatrix}$$

$$+ \frac{1}{81}\begin{bmatrix} 27 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 10 & 6 & 0 & 8 \\ 0 & 6 & 3 & 0 & 0 & 12 & 6 & 0 \\ 0 & 10 & 6 & 3 & 4 & 0 & 0 & 4 \\ 0 & 0 & 6 & 0 & 5 & 0 & 12 & 4 \\ 0 & 0 & 12 & 6 & 2 & 5 & 0 & 2 \\ 0 & 4 & 0 & 10 & 6 & 2 & 5 & 0 \\ 0 & 4 & 0 & 8 & 0 & 2 & 4 & 9 \end{bmatrix} = \frac{1}{81}\left[\begin{array}{c|ccc|ccc|c} 81 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 9 & 10 & 12 & 16 & 6 & 16 & 12 \\ 0 & 12 & 9 & 10 & 6 & 16 & 16 & 12 \\ 0 & 10 & 12 & 9 & 16 & 16 & 6 & 12 \\ \hline 0 & 6 & 16 & 16 & 15 & 6 & 16 & 6 \\ 0 & 16 & 16 & 6 & 16 & 15 & 6 & 6 \\ 0 & 16 & 6 & 16 & 6 & 16 & 15 & 6 \\ \hline 0 & 12 & 12 & 12 & 6 & 6 & 6 & 27 \end{array}\right]$$

Notice, however, that none of the three individual components $[\mathsf{Par}, G_{3,\frac{2}{3}}, \pi']$ for $\pi' \in \mathcal{P}_{C_3}$ are the transpose of any of the components $[\mathsf{Par}, G_{3,\frac{2}{3}}, \pi]$, where $\pi \in \mathcal{P}_{C_3}$.

Finally, consider the update order distribution generated by S_3, the full automorphism group of K_3. We get

$$[\mathsf{Par}, G_{3,\frac{2}{3}}, \mathcal{P}_{S_3}] = \frac{1}{2}[\mathsf{Par}, G_{3,\frac{2}{3}}, \mathcal{P}_{C_3}] + \frac{1}{2}[\mathsf{Par}, G_{3,\frac{2}{3}}, \mathcal{P}'_{C_3}]$$

$$= \frac{1}{162}\left[\begin{array}{c|ccc|ccc|c} 162 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 18 & 22 & 22 & 22 & 22 & 32 & 24 \\ 0 & 22 & 18 & 22 & 22 & 32 & 22 & 24 \\ 0 & 22 & 22 & 18 & 32 & 22 & 22 & 24 \\ \hline 0 & 22 & 22 & 32 & 30 & 22 & 22 & 12 \\ 0 & 22 & 32 & 22 & 22 & 30 & 22 & 12 \\ 0 & 32 & 22 & 22 & 22 & 22 & 30 & 12 \\ \hline 0 & 24 & 24 & 24 & 12 & 12 & 12 & 54 \end{array}\right],$$

and as guaranteed by Proposition 5.8, the induced permutations $(1\ 2)(5\ 6)$, $(2\ 3)(4\ 5)$, and $(1\ 3)(4\ 6)$ all preserve the Markov chain, and thus

$$S_3 \hookrightarrow \mathsf{Stab}[\mathsf{Par}, G_{3,\frac{2}{3}}, \mathcal{P}_{S_3}] \cong S_3\ .$$

None of the results in this section should be too surprising, however, they form a good starting point for the study of stochastic SDSs.

6. Conclusions and Future Research

One of the underlying themes throughout this dissertation, though it was not explicitly stated, was the sensitivity of some aspect of the dynamics of an SDS to changes in the update order. When studying large complex systems it can be desirable to understand *update order stability.* One way to study this is to see how adding or removing edges affects the stability, or said differently, whether the dynamics are, in general, more stable over sparse graphs or over dense graphs.

In Sections 2 and 3, the idea of update order stability was analyzed by defining different notions of equivalence, which captured certain key properties of the phase spaces of the SDS maps under variations of the update order. We constructed neutral networks for functional and cycle equivalence, and the number of connected components of these objects gave an upper bound for the number of SDS maps up to equivalence obtainable by changing the update order for a fixed choice of functions. In both cases, adding edges to the dependency graph increases these upper bounds, and one of these bounds is known to be sharp. One or both of these notions of equivalence can be used to characterize the stability of an SDS with respect to update order. For example, if Y is a connected tree on n-vertices, then there are 2^{n-1} SDS maps of the form $[\mathsf{Nor}_Y, \pi]$ for $\pi \in S_Y$. However, if $Y = K_n$, then there are $n!$ such maps. This can lead one to the conclusion that in general, *adding edges to the dependency graph of an* SDS *causes the dynamics to become less stable with respect to changes in the update order.*

One of the prevailing themes of Section 4 was word-independent SDSs. This idea can be extended to a measure of update order stability. For a fixed choice of functions $\mathfrak{F}_Y$, one can compute the number of states that are periodic under the map $[\mathfrak{F}_Y, \pi]$ for *all* $\pi \in S_Y$, and the number of states that are periodic for $[\mathfrak{F}_Y, \pi]$ for *some* $\pi \in S_Y$. The quotient of these two numbers is a positive number not greater than than 1, and can be used as a measure of update order stability. The most stable SDSs according to this measure are the word-independent systems, because the sets of periodic points are unchanged under variations to the update order. The observation that all invertible SDS maps and all fixed point systems are word-independent, as well as the suggested correlation between the (normalized) signature and being word-independent, leads one to hypothesize that word-independence, and this measure of update order stability, is more a property of the functions rather than the dependency graph. A case could thus be made that *adding edges to the dependency graph of an* SDS *has little or no effect on the stability of the dynamics with respect to changes in the update order.*

We conclude our discussion of update order stability by stating a recent theorem from [24]. In this paper, the authors consider SDSs induced by threshold functions, which arise frequently in modeling epidemics [22] or biological networks [21]. They discuss a function of an SDS which captures a different aspect about update order stability. A vertex function $f\colon \mathbb{F}_2^{d+1} \to \mathbb{F}_2$ is a *k-threshold function* if $f(x) = 1$ if and only if at least k entries of x are 1. It is clear that threshold functions are fixed-point systems, by virtue of being one-way functions. The main question in [24] is to understand how many different fixed points a state in $\mathbb{F}_2^n$ can reach under a threshold SDS map by varying the update order. To do this the *ω-limit set* of a map $\phi\colon K^n \to K^n$ is defined to be the set of all periodic points $\mathbf{z} \in K^n$ such that $\phi^m(\mathbf{y}) = \mathbf{z}$ for some $m \geq 0$. For a given SDS $(Y, \mathfrak{F}_Y, \pi)$ and state $\mathbf{y} \in \mathbb{F}_2^n$, the ω-limit set of the SDS map $[\mathfrak{F}_Y, \pi]$ is denoted by $\omega_\pi(\mathbf{y})$. If $\mathcal{P} \subset S_Y$ is a collection of simple update orders, then the *ω-limit set of* $\mathbf{y}$ *with respect to* $\mathcal{P}$ is defined to be

$$\omega_{\mathcal{P}}(\mathbf{y}) = \bigcup_{\pi \in \mathcal{P}} \omega_\pi(\mathbf{y}) \; .$$

If there are large periodic cycles in the phase space of the SDS, then the size of $\omega_{\mathcal{P}}(\mathbf{y})$ is not too insightful. However, if $[\mathfrak{F}_Y, \pi]$ contains only fixed points, then $\omega_{\mathcal{P}}(\mathbf{y})$ is a direct measure of stability that describes how many fixed points can be reached from $\mathbf{y}$ by varying the update order according to $\mathcal{P}$. For a set of functions $\mathfrak{F}_Y$, define

$$\omega(\mathfrak{F}_Y) = \max |\{\omega_{S_Y}(\mathbf{y}) \mid \mathbf{y} \in K^n\}| \ ,$$

and this captures the maximum possible number of periodic points that can be reached from any state by variation of the update order. Again, this measure loses its value for non-fixed point systems. The following result is proven in [24].

Theorem 6.1. *Let $\mathfrak{f}_Y$ be a sequence of 2-threshold vertex functions. If $Y = \mathsf{Star}_n$ then $\omega(\mathfrak{F}_Y) = 2^n - n$. If $Y = K_n$, then $\omega(\mathfrak{F}_Y) = n + 1$.*

Additionally, this is extended to the random graph model $G_{n,p}$ for various values of p. The main result is that for sparse graphs, $\omega(\mathfrak{F}_Y) = \Theta(2^n)$ (with sufficiently high probability), but for dense graphs, $\omega(\mathfrak{F}_Y) = \Theta(n)$. Thus, using this notion of update order stability, one might be led to conclude that *adding edges to the dependency graph of an* SDS *causes the dynamics to become more stable with respect to changes in the update order.*

The moral here is that one must be careful when making general statements about update order stability, because there are numerous ways to define it, and two proposed measures can be uncorrelated, or even inversely correlated. In conclusion, we have seen three natural ways to characterize update order stability of an SDS $(Y, \mathfrak{F}_Y, \pi)$, namely by computing:

- the number of SDS maps $[\mathfrak{F}_Y, \pi]$ up to functional (or cycle) equivalence for $\pi \in S_Y$,

- the percentage of states that are periodic for an SDS $[\mathfrak{F}_Y, \pi]$ for some $\pi \in S_Y$, that are additionally periodic for every $\pi \in S_Y$,
- The maximum number of fixed points that can be reached from a given state $\mathbf{y}$ under the map $[\mathfrak{F}_Y, \pi]$ over all $\pi \in S_Y$.

Depending on which of these measures are studied, one can make a case that the stability of the dynamics of an SDS is directly related, inversely related, or completely unrelated, to the density of edges in the dependency graph. We bring this up, because besides being interesting in its own right, it should serve as a cautionary warning for someone trying to study and characterize update order stability. These three notions have only been studied independently. Perhaps there are ways to draw connections between them to paint a more complete picture. This could be the topic of an excellent research project, and is an interesting thought on which to conclude this dissertation.

Periodic points	Wolfram rules
$\{\mathbf{0}\}$	0, 8, 64, 32, 40, 96
$\{\mathbf{1}\}$	235, 239, 249, 251, 253, 255
$\{\mathbf{0},\mathbf{1}\}$	128, 136, 152, 160, 168, 184, 188, 192, 194 224, 226, 230, 234, 238, 248, 250, 252, 254
{No '11', '000'}	5, 13, 69
{No '00', '111'}	79, 93, 95
{No '11', '000'} $\cup \{\mathbf{1}\}$	133, 141, 197
{No '00', '111'} $\cup \{\mathbf{0}\}$	78, 92, 94
{No '11', '010'}	72
{No '00', '101'}	237
{No '11', '101'} $\cup \{\mathbf{1}\}$	164, 172, 228
{No '00', '010'} $\cup \{\mathbf{0}\}$	202, 216, 218
{No '111', '000', '1100'}	29
{No '111', '000', '0011'}	71
{No '111', '000', '1100'} $\cup \{\mathbf{0}\}$	28
{No '111', '000', '0011'} $\cup \{\mathbf{1}\}$	199
{No '111', '000', '0011'} $\cup \{\mathbf{0}\}$	70
{No '111', '000', '1100'} $\cup \{\mathbf{1}\}$	157
{No '111', '000'}	77
{No '101', '010'}	232
{No '11'}	1, 4, 9, 12, 65, 68
{No '00'}	111, 125, 127, 207, 221, 223
{No '11'} $\cup \{\mathbf{1}\}$	129, 132, 137, 140, 193, 196
{No '00'} $\cup \{\mathbf{0}\}$	110, 124, 126, 206, 220, 222
{No '101'} $\cup \{\mathbf{0}\}$	236
{No '010'} $\cup \{\mathbf{1}\}$	200
{No '111'}	73, 76
{No '000'}	109, 205
$\mathbb{F}_2^n$	51, 54, 57, 60, 99, 102, 105, 108 147, 150, 153, 156, 195, 198, 201, 204

Table 8. Periodic points of word-independent ACAs, indexed by Wolfram rules

Rule	tag	inv	refl	i.r.	flips	sig	$m_{i,i+1}$	G	Per
0	`0000`	255	0	255	4	-4	1	1	$\{\mathbf{0}\}$
1	`x000`	127	1	127	5	-3	6	$S_{L(n)}$ or $A_{L(n)}$	$P_{n,7}$
105	`xx--`	105	105	105	4	0	3	?	$\mathbb{F}_2^n$
126	`-x11`	129	126	129	4	2	6	$S_{L(n)}$ or $A_{L(n)}$	$\bar{P}_{n,7} \cup \{\mathbf{0}\}$
128	`0-00`	254	128	254	3	-3	1	1	$\{\mathbf{0},\mathbf{1}\}$
150	`--xx`	150	150	150	4	0	3	?	$\mathbb{F}_2^n$
232	`01--`	232	232	232	2	0	1	1	$P_{n,6}$
4	`-000`	223	4	223	3	-3	1	1	$P_{n,7}$
5	`1000`	95	5	95	4	-2	1	1	$P_{n,1}$
32	`0x00`	251	32	251	5	-3	1	1	$\{\mathbf{0}\}$
51	`xxxx`	51	51	51	8	0	2	C_2^n	$\mathbb{F}_2^n$
54	`-xxx`	147	54	147	6	0	12	A_{n-1}	$\mathbb{F}_2^n$
72	`00--`	237	72	237	2	-2	1	1	$P_{n,2}$
73	`x0--`	109	73	109	3	-1	6	?	$P_{n,9}$
76	`-0--`	205	76	205	1	-1	1	1	$P_{n,9}$
77	`10--`	77	77	77	2	0	1	1	$P_{n,5}$
94	`-011`	133	94	133	3	1	1	1	$\bar{P}_{n,1} \cup \{\mathbf{0}\}$
108	`-x--`	201	108	201	2	0	6	?	$\mathbb{F}_2^n$
132	`--00`	222	132	222	2	-2	1	1	$P_{n,7} \cup \{\mathbf{1}\}$
160	`0100`	250	160	250	4	-2	1	1	$\{\mathbf{0},\mathbf{1}\}$
164	`-100`	218	164	218	3	-1	1	1	$P_{n,3} \cup \{\mathbf{1}\}$
200	`0---`	236	200	236	1	-1	1	1	$\bar{P}_{n,8}$
204	`----`	204	204	204	0	0	1	1	$\mathbb{F}_2^n$
8	`00-0`	239	64	253	3	-3	1	1	$\{\mathbf{0}\}$
9	`x0-0`	111	65	125	4	-2	6	$S_{L(n)}$ or $A_{L(n)}$	$P_{n,7}$
12	`-0-0`	207	68	221	2	-2	1	1	$P_{n,7}$
13	`10-0`	79	69	93	3	-1	1	1	$P_{n,1}$
28	`-0-x`	199	70	157	3	-1	2	C_2^n	$P_{n,4} \cup \{\mathbf{0}\}$
29	`10-x`	71	71	29	4	0	2	C_2^n	$P_{n,4}$
40	`0x-0`	235	96	249	4	-2	1	1	$\{\mathbf{0}\}$
57	`xx-x`	99	99	57	6	0	12	A_n	$\mathbb{F}_2^n$
60	`-x-x`	195	102	153	4	0	4	$GL(n,2)$	$\mathbb{F}_2^n$
78	`-01-`	141	92	197	2	0	1	1	$\{\mathbf{0}\}$
110	`-x1-`	137	124	193	3	1	6	$S_{L(n)}$ or $A_{L(n)}$	$\bar{P}_{n,7} \cup \{\mathbf{0}\}$
136	`0--0`	238	192	252	2	-2	1	1	$\{\mathbf{0},\mathbf{1}\}$
140	`---0`	206	196	220	1	-1	1	1	$P_{n,7} \cup \{\mathbf{1}\}$
152	`0--x`	230	194	188	3	-1	1	1	$\{\mathbf{0},\mathbf{1}\}$
156	`---x`	198	198	156	2	0	6	?	$\mathbb{F}_2^n$
168	`01-0`	234	224	248	3	-1	1	1	$\{\mathbf{0},\mathbf{1}\}$
172	`-1-0`	202	228	216	2	0	1	1	$P_{n,3}$
184	`01-x`	226	226	184	4	0	1	1	$\{\mathbf{0},\mathbf{1}\}$

TABLE 9. Summary of the 104 word-independent Wolfram rules, up to equivalence, and grouped into three classes containing the (*i*) symmetric rules, (*ii*) quasi-symmetric rules, and (*iii*) remaining rules.

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