Combinatorics of Discrete Dynamical Systems and Coxeter Theory

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Outline

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Sequential dynamical systems

A sequential dynamical system (SDS) is a triple consisting of:

- A graph $Y$ with vertex set $\mathcal{V}(Y) = \{1, 2, \ldots, n\}$.
- For each vertex $i$ a state $y_i \in K$ (e.g. $\mathbb{F}_2 = \{0, 1\}$) and a local function $F_i : K^n \rightarrow K^n$
  $$F_i(y = (y_1, y_2, \ldots, y_n)) = (y_1, \ldots, \underbrace{f_i(y[i]), y_{i+1}, \ldots, y_n}_{\text{vertex function}}) .$$
- A ordering $\pi = \pi_1 \pi_2 \cdots \pi_n \in S_Y$ of the vertex set.

The SDS map generated by the triple $(Y, (F_i)_1^n, \pi)$ is

$$\hat{\mathfrak{A}}_{Y, \pi} = F_{\pi_n} \circ F_{\pi_{n-1}} \circ \cdots \circ F_{\pi_1}.$$
Question: What does it mean for two SDSs to be “equivalent”?

Definition

Two SDSs are **functionally equivalent** if their SDS maps are identical as functions $K^n \rightarrow K^n$.

Definition

Two finite dynamical systems $\phi, \psi: K^n \rightarrow K^n$ are **dynamically equivalent** if there is a bijection $h: K^n \rightarrow K^n$ such that

$$\psi \circ h = h \circ \phi.$$  

(i.e., phase spaces are isomorphic).

Definition

Two finite dynamical systems $\phi, \psi: K^n \rightarrow K^n$ are **cycle equivalent** if there exists a bijection $h: \text{Per}(\phi) \rightarrow \text{Per}(\psi)$ such that

$$\psi|_{\text{Per}(\psi)} \circ h = h \circ \phi|_{\text{Per}(\phi)}.$$  

(i.e., phase spaces are isomorphic when restricted to the periodic points).
Example. Define the function $\nor_k : \mathbb{F}_2^k \longrightarrow \mathbb{F}_2$ by $\nor_k(x) = \prod_{i=1}^{k}(1 + x_i)$.

$\mathbf{[\text{Nor}_{\text{Circ}_4}, \pi]}$ for given update sequences:

We will establish later that these are the only phase spaces up to isomorphism obtainable by varying the update sequence.
Questions on Equivalence of SDSs

- *What dynamical properties are preserved as the update sequence changes?*

- How many SDS maps up to equivalence are obtainable by varying the update sequence?

- Can we characterize equivalence combinatorially, through properties of the base graph?

*Key observation:* We can associate an update sequence $\pi \in S_Y$ with an acyclic orientation, $O_Y^\pi \in \text{Acyc}(Y)$, by a well-defined map

$$f_Y : S_Y \longrightarrow \text{Acyc}(Y), \quad f_Y(\pi) = O_Y^\pi,$$

where $\pi$ is a linear extension of $O_Y^\pi$.

Explicitly, if $\pi = \pi_1\pi_2\cdots\pi_n$ then $\{i,j\} \in e[Y]$ is oriented $(i,j)$ iff $i$ appears before $j$ in $\pi$.

- For any $\pi, \sigma \in S_Y$, define $\pi \sim_\alpha \sigma$ iff $f_Y(\pi) = f_Y(\sigma)$. This is an equivalence relation on $S_Y$. 
Equivalence on acyclic orientations

- For $O, O' \in \text{Acyc}(Y)$ define $O \sim_{\alpha} O'$ if $O' = \gamma(O)$ for some $\gamma \in \text{Aut}(Y)$.

A cyclic 1-shift (left) of a linear extension of $O_Y$ corresponds to converting a source of $O_Y$ into a sink.

- This source-to-sink operation (or a “click”) puts an equivalence relation on $\text{Acyc}(Y)$, denoted $\sim_{\kappa}$.

![Source-to-sink operations](image)

Figure: Source-to-sink operations

- $\text{Aut}(Y)$-actions, with source-to-sink operations, together yield a coarser equivalence relation $\sim_{\kappa}$.
Equivalence on update sequences

Let $S_Y$ denote the set of permutation update sequences of $v[Y]$.

- $\pi \sim_\alpha \sigma \iff O^\pi_Y = O^\sigma_Y$.
- $\pi \sim_{\bar{\alpha}} \sigma \iff O^\pi_Y \sim_{\bar{\alpha}} O^\sigma_Y$ (related by $\gamma \in \text{Aut}(Y)$).
- $\pi \sim_\kappa \sigma \iff O^\pi_Y \sim_\kappa O^\sigma_Y$ (related by source-to-sink moves).
- $\pi \sim_{\bar{\kappa}} \sigma \iff O^\pi_Y \sim_{\bar{\kappa}} O^\sigma_Y$ (related by $\gamma \in \text{Aut}(Y)$ & source-to-sink moves).

**Theorem**

Let $\mathcal{F}_Y$ be a sequence of $\text{Aut}(Y)$-invariant functions.

- If $\pi \sim_\alpha \sigma$, then $[\mathcal{F}_Y, \pi]$ and $[\mathcal{F}_Y, \sigma]$ are functionally equivalent.
- If $\pi \sim_{\bar{\alpha}} \sigma$, then $[\mathcal{F}_Y, \pi]$ and $[\mathcal{F}_Y, \sigma]$ are dynamically equivalent.
- If $\pi \sim_\kappa \sigma$, then $[\mathcal{F}_Y, \pi]$ and $[\mathcal{F}_Y, \sigma]$ are cycle equivalent.
- If $\pi \sim_{\bar{\kappa}} \sigma$, then $[\mathcal{F}_Y, \pi]$ and $[\mathcal{F}_Y, \sigma]$ are cycle equivalent.
**Enumeration problems**

- \( \alpha(Y) := \text{Acyc}(Y) = T_Y(2, 0) \) satisfies
  \[
  \alpha(Y) = \alpha(Y/e) + \alpha(Y \setminus e) \quad \text{for any edge } e
  \]

- \( \tilde{\alpha}(Y) := \text{Acyc}(Y)/\sim_{\tilde{\alpha}} = \frac{1}{\text{Aut}(Y)} \sum_{\gamma \in \text{Aut}(Y)} \alpha(\langle \gamma \rangle \setminus Y) \)

- \( \kappa(Y) := \text{Acyc}(Y)/\sim_{\kappa} = T_Y(1, 0) \) satisfies
  \[
  \kappa(Y) = \kappa(Y/e) + \kappa(Y \setminus e) \quad \text{for any cycle edge } e
  \]

- \( \tilde{\kappa}(Y) := \text{Acyc}(Y)/\sim_{\tilde{\kappa}} = \frac{1}{\text{Aut}(Y)} \sum_{\gamma \in \text{Aut}(Y)} |\text{Fix}(\gamma)| \)

*But what is \(|\text{Fix}(\gamma)| \)??
An example

Let $Q_2^3$ be the binary 3-cube. A tedious calculation gives $\alpha(Y) = 1862$.

$$
\kappa\left(\begin{array}{c}
\vdots
\end{array}\right) = \kappa\left(\begin{array}{c}
\vdots
\end{array}\right) + \kappa\left(\begin{array}{c}
\vdots
\end{array}\right) + \kappa\left(\begin{array}{c}
\vdots
\end{array}\right) + 2\kappa\left(\begin{array}{c}
\vdots
\end{array}\right) + \kappa\left(\begin{array}{c}
\vdots
\end{array}\right)
$$

$$
= \kappa\left(\begin{array}{c}
\vdots
\end{array}\right) + 2\kappa\left(\begin{array}{c}
\vdots
\end{array}\right) + 2\kappa\left(\begin{array}{c}
\vdots
\end{array}\right) + \kappa\left(\begin{array}{c}
\vdots
\end{array}\right) + \kappa\left(\begin{array}{c}
\vdots
\end{array}\right)
$$

$$
= \kappa\left(\begin{array}{c}
\vdots
\end{array}\right) + 4\kappa\left(\begin{array}{c}
\vdots
\end{array}\right) + 2\kappa\left(\begin{array}{c}
\vdots
\end{array}\right) + \kappa\left(\begin{array}{c}
\vdots
\end{array}\right) + \kappa\left(\begin{array}{c}
\vdots
\end{array}\right)
$$

$$
= 27 + 64 + 16 + 12 + 14 = 133.
$$

► In summary, we have:

$$
\alpha(Q_2^3) = 1862, \quad \bar{\alpha}(Q_2^3) = 54, \quad \kappa(Q_2^3) = 133, \quad \delta(Q_2^3) = 67, \quad \bar{\kappa}(Q_2^3) = \bar{\delta}(Q_2^3) = 8.
$$

► If $Y = Q_2^3$, then for a fixed choice of functions $\mathfrak{F}_Y$, there are at most 8 possible cycle structures of the SDS map $[\mathfrak{F}_Y, \pi]$, up to isomorphism.
\( \kappa(Y) \) for some special graph classes

**Proposition ([6])**

For \( v \in v[Y] \), let \( |\text{Acyc}_v(Y)| \) be the number of acyclic orientations of \( Y \) where \( v \) is the unique source. There is a bijection

\[
\phi_v : \text{Acyc}_v(Y) \longrightarrow \text{Acyc}(Y) / \sim_\kappa.
\]

**Corollary**

For any vertex \( v \) of \( Y \) the set \( \text{Acyc}_v(Y) \) is a transversal of \( \text{Acyc}(Y) / \sim_\kappa \).

- If \( Y \) is a tree, then \( \kappa(Y) = 1 \).
- If \( Y \) is an \( n \)-cycle, then \( \kappa(Y) = n - 1 \).
- If \( Y \oplus v \) is the vertex join of \( Y \), then \( \kappa(Y \oplus v) = \alpha(Y) \).
- \( \kappa(K_n) = (n - 1)! \).
The $\nu$-invariant

Let $P = v_1v_2\cdots v_k$ be a path in $Y$. Define $\nu_P(O_Y)$ to be the number of edges oriented $(v_i, v_{i+1})$, minus the number of edges oriented $(v_{i+1}, v_i)$.

*Easy fact:* If $P$ is a cycle, then $\nu_P(O_Y)$ is invariant under clicks.

Let $Y = \text{Circ}_n$, and let $P$ traverse $Y$ once. The possible values for $\nu_P(\text{Circ}_n)$ are $\pm(n - 2), \pm(n - 4), \pm(n - 6), \ldots$. Therefore, $\kappa(\text{Circ}_n) \geq n - 1$.

By the recurrence $\kappa(Y) = \kappa(Y/e) = \kappa(Y \setminus e)$, and with base case $\kappa(\text{Tree}) = 1$, we get $\kappa_P(\text{Circ}_n) = n - 1$.

Therefore, $\nu$ is a complete invariant of $\text{Acyc}(\text{Circ}_n) / \sim_\kappa$, i.e., if $Y = \text{Circ}_n$,

$$\nu_P(O_Y) = \nu_P(O'_Y) \iff O_Y \sim_\kappa O'_Y.$$
The $\nu$-invariant (cont.)

\[ \nu = 4 \quad \nu = 2 \quad \nu = 0 \quad \nu = -2 \quad \nu = -4 \]

Figure: A transversal for Acyc(Circ$_6$)/$\sim_\kappa$.

In fact, taken over all cycles, $\nu$ is a complete invariant of Acyc($Y$)/$\sim_\kappa$:

**Theorem (M–, Mortveit [7])**

If $\nu_C(O_Y) = \nu_C(O'_Y)$ for every cycle $C$ in $Y$, then $O_Y \sim_\kappa O'_Y$. 
Coxeter groups

Definition

A Coxeter group is a group with presentation
\[ \langle s_1, \ldots, s_n \mid s_i s_j^{m_{ij}} \rangle \]
where \( m_{ij} > 1 \) iff \( i \neq j \).

It follows easily that \( |s_i| = 2 \), and \( |s_i s_j| = |s_j s_i| \).

Think of a Coxeter group as a generalized reflection group (more on this later).

Recall, for any non-zero vectors \( \mathbf{v}, \mathbf{w} \in \mathbb{R}^n \), the reflection of \( \mathbf{v} \) across the hyperplane orthogonal to \( \mathbf{w} \) is
\[ \mathbf{v} - 2 \frac{\langle \mathbf{v}, \mathbf{w} \rangle}{\langle \mathbf{w}, \mathbf{w} \rangle} \mathbf{w}. \]
Coxeter graphs and acyclic orientations

A Coxeter system is a triple \((W, S, \Gamma)\) where \(W\) is a Coxeter group, \(S\) is the set of reflections that generate \(W\), and \(\Gamma\) is the Coxeter graph:

\[
v[\Gamma] = S, \quad e[\Gamma] = \{\{s_i, s_j\} \mid m_{ij} \geq 3\}.
\]

Additionally, each edge \(\{s_i, s_j\}\) is labeled with \(m_{ij}\) (usually the label is omitted for \(m_{ij} = 3\) because these are the most common).

**Note:** Edges correspond to non-commuting pairs of reflections.

- A Coxeter element is the product of the generators in any order.

- There is a bijection between the set of Coxeter elements \(C := C(W, S, \Gamma)\) and \(\text{Acyc}(\Gamma)\) (see [11]).
Conjugacy of Coxeter elements

- Conjugating a Coxeter element by a simple reflection cyclically shifts the word, and corresponds to a **source-to-sink** operation (or “click”):

\[
\begin{align*}
  s_{\pi(1)}(s_{\pi(1)}s_{\pi(2)}\cdots s_{\pi(n)})s_{\pi(1)} &= s_{\pi(2)}s_{\pi(3)}\cdots s_{\pi(n)}s_{\pi 1} .
\end{align*}
\]

Therefore, the equivalence relation \( \sim_\kappa \) carries over to \( C(W, S, \Gamma) \).

- Clearly, if \( c \sim_\kappa c' \), then \( c \) and \( c' \) are conjugate in \( W \).

- Therefore, \( \kappa(\Gamma) \) is an upper bound on the number of conjugacy classes of Coxeter elements [6].

**Open question:** Is this bound sharp, i.e., does the converse of the statement above hold?
Conjugacy in simply-laced Coxeter groups

A Coxeter system is *simply-laced* if $m_{ij} \leq 3$.

**Theorem (H. Eriksson, 1994 [2])**

Let $(W, S, \Gamma)$ be a simply-laced Coxeter system where $\Gamma = \text{Circ}_n$ (i.e., $W = \tilde{A}_{n-1}$ is the affine Weyl group). Then two Coxeter elements $c, c' \in C(W, S, \Gamma)$ are conjugate if and only if $c \sim_{\kappa} c'$.

**Theorem (J.-Y. Shi, 2001 [12])**

Let $(W, S, \Gamma)$ be a simply-laced Coxeter system where $\Gamma$ is unicyclic. Then two Coxeter elements $c, c' \in C(W, S, \Gamma)$ are conjugate if and only if $c \sim_{\kappa} c'$.

**Theorem (M-. Mortveit, 2008 [7])**

Let $(W, S, \Gamma)$ be a simply-laced Coxeter system. Then two Coxeter elements $c, c' \in C(W, S, \Gamma)$ are conjugate if and only if $c \sim_{\kappa} c'$.
Natural reflection representation

Define \( a_{i,j} = \cos \frac{\pi}{m_{ij}} \).

The natural reflection representation of \( W \) is defined on the generators \( s \in S \) by

\[
s_i \mapsto I_n - 2E_{i,i} + \sum_{j : m_{ij} \geq 3} a_{i,j} E_{i,j}.
\]

Example.

\[
\begin{pmatrix}
1 & \cdots & 1 \\
\vdots & \ddots & \vdots \\
1 & \cdots & a_i, i-1, i-1, a_i, i, a_i, i+1, \cdots \\
\end{pmatrix}
\]
Spectral classes

Identify $w \in W$ with the corresponding linear transformation in the natural reflection representation.

If $w$ and $w'$ are conjugate in $W$, then they have the same spectral class.

*Question* [12]: Given a Coxeter system $(W, S, \Gamma)$, how many spectral classes do the Coxeter elements in $C(W, S, \Gamma)$ fall into?

Two $\kappa$-classes that have respective acyclic orientations $O_\Gamma$ and $O'_\Gamma$ such that $\varphi: O_\Gamma \leftrightarrow O'_\Gamma$ for some $\varphi \in \text{Aut}(\Gamma)$ also have the same spectral class.

$\blacktriangleright$ Therefore, $\bar{\kappa}(\Gamma)$ is an upper bound for the number of spectral classes.
An example

Let $\Gamma = K_{2,3}$, with vertex set $\{1, 3, 5\} \sqcup \{2, 4\}$.

$\alpha(\Gamma) = 46$, $\kappa(\Gamma) = 7$, and $\bar{\kappa}(\Gamma) = 2$. There are 2 spectral classes (See Shi, 2001 [12]):

\[
\{12345, 23451, 52341, 51234, 45123, 34512\} \\
\{12543, 25431, 32541, 31254, 43125, 54312\} \\
\{32145, 35214, 52143, 21435, 14352, 43521\} \\
\{14523, 45231, 34512, 31452, 23145, 52314\} \\
\{14325, 43251, 54321, 51432, 25143, 32514\} \\
\{34124, 41235, 54123, 35412, 23541, 12354\} \\
\{24351, 21354, 13524, 41352, 52431, 15243, 12435, 31245, 32451, 35241\}
\]

Elements in the first six classes have characteristic polynomial

\[ f(x) = x^5 - 3x^4 - 6x^3 - 6x^2 - 3x + 1. \]

Elements in the last class have characteristic polynomial \( f(x) = x^5 - x^4 - 8x^3 - 8x^2 - x + 1. \)
An example (cont.)

Figure: The update graph $U(K_{2,3})$: Connected components are in 1–1 correspondence with $\text{Acyc}(K_{2,3})$.

Consider the mapping $(s_{\pi_i})_i \xrightarrow{\phi} (\pi_i \mod 2)_i$.

Non-adjacency in $\Gamma$ coincides with parity, that is, if $c = c'$, then $\phi(c) = \phi(c')$.

- 12 size-1 components: 10101
- 24 size-2 components: 01011, 11010, 01101, 10110.
- 6 size-4 components: 10011, 11001.
- 2 size-6 components: 01110
- 2 size-12 components: 11100, 00111.
Component at left: $\phi(\pi) \in \{01101, 11010, 10101, 01011, 10110\}$.

Component at right: $\phi(\pi) \in \{11100, 11001, 10011, 00111, 01110\}$.

Figure: The graph $C(K_{2,3})$ contains the component on the left, and three isomorphic copies of the structure on the right (but with different vertex labels).
Quiver representations [8]

A quiver is a finite directed graph (loops and multiple edges allowed).

A quiver $Q$ with a field $K$ gives rise to a path algebra $KQ$.

There is a natural correspondence (categorial equivalence) between $KQ$-modules, and $K$-representations of $Q$.

- A path algebra is finite-dimensional if and only if the quiver is acyclic. Modules over finite-dimensional path algebras form a reflective subcategory.

- A reflection functor maps representations of a quiver $Q$ to representations of a quiver $Q'$, where $Q'$ differs from $Q$ by a source-to-sink operation.

- A composition of $n = |v(Q)|$ distinct reflection functors is not the identity, but a Coxeter functor.
Node-firing games [3]

- In the *chip-firing game*, each vertex of a graph is given some number (possibly zero) of chips.

If vertex $i$ has degree $d_i$, and at least $d_i$ chips, then a legal move (or a “click”) is a transfer of one chip to each neighbor.

A legal move is in a sense a generalization of a source-to-sink operation.

- In the *numbers game*, each vertex of a graph is assigned an integer value, and the edges are weighted according to the $m_{ij}$ relations of the Coxeter group.

The legal sequences of moves in the numbers game are in 1–1 correspondence with the reduced words of the Coxeter group with that Coxeter graph.
## Summary of SDS / Coxeter theory connections

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Summary of future research

**Combinatorics**

- Is there a nice closed-form or easily computable solution to $\bar{\kappa}(\Gamma)$?

**Sequential dynamical systems**

- Are $\bar{\kappa}(Y)$ and $\bar{\delta}(Y)$ sharp upper bounds for the number of SDS maps up to cycle equivalence?

**Coxeter groups**

- Prove that two Coxeter elements are conjugate iff they are $\kappa$-equivalent, for a non-simply-laced Coxeter system.

- Is $\bar{\kappa}(\Gamma)$ a sharp upper bound for the number of spectral classes of Coxeter elements of $(W, S, \Gamma)$? If not, for which graphs does it fail, and by how much?
References


