Combinatorics of Discrete Dynamical Systems and Coxeter Theory

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Outline

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   - Equivalence of dynamics
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   - Connections to other areas of mathematics
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Sequential dynamical systems

- A *sequential dynamical system* (SDS) is a triple consisting of:
  - A graph Y with vertex set $\mathrm{v}[Y] = \{1, 2, \ldots, n\}$.
  - For each vertex i a state $y_i \in K$ (e.g. $\mathbb{F}_2 = \{0, 1\}$) and a local function $F_i \colon K^n \longrightarrow K^n$

$$F_i(\mathbf{y} = (y_1, y_2, \ldots, y_n)) = (y_1, \ldots, y_{i-1}, \underbrace{f_i(\mathbf{y}[i])}_{\text{vertex function}}, y_{i+1}, \ldots, y_n) \ .$$

  - A ordering $\pi = \pi_1 \pi_2 \cdots \pi_n \in S_Y$ of the vertex set.
- The SDS map generated by the triple $(Y, (F_i)_1^n, \pi)$ is

$$\boxed{[\mathfrak{F}_Y, \pi] = F_{\pi_n} \circ F_{\pi_{n-1}} \circ \cdots \circ F_{\pi_1} \ .}$$

- Question: *What does it mean for two* SDSs *to be "equivalent"?*

Definition

Two SDSs are *functionally equivalent* if their SDS maps are identical as functions $K^n \longrightarrow K^n$.

Definition

Two finite dynamical systems $\phi, \psi \colon K^n \longrightarrow K^n$ are *dynamically equivalent* if there is a bijection $h \colon K^n \longrightarrow K^n$ such that

$$\psi \circ h = h \circ \phi \, .$$

(i.e., phase spaces are isomorphic).

Definition

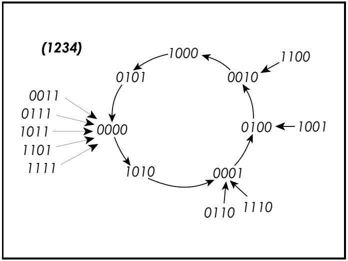
Two finite dynamical systems $\phi, \psi \colon K^n \to K^n$ are *cycle equivalent* if there exists a bijection $h \colon \mathrm{Per}(\phi) \longrightarrow \mathrm{Per}(\psi)$ such that

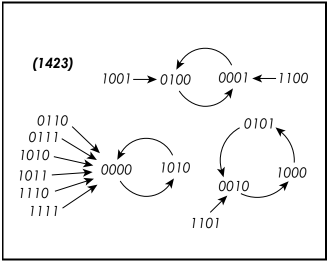
$$\psi|_{\mathrm{Per}(\psi)} \circ h = h \circ \phi|_{\mathrm{Per}(\phi)} \, .$$

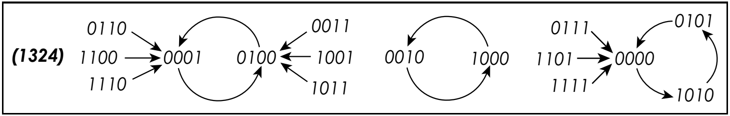
(i.e., phase spaces are isomorphic when restricted to the periodic points).

Example. Define the function $\mathrm{nor}_k \colon \mathbb{F}_2^k \longrightarrow \mathbb{F}_2$ by $\mathrm{nor}_k(\mathbf{x}) = \prod_{i=1}^{k}(1 + x_i)$.

▶ $[\mathbf{Nor}_{\mathrm{Circ}_4}, \pi]$ for given update sequences:







We will establish later that these are the *only* phase spaces up to isomorphism obtainable by varying the update sequence.

Questions on Equivalence of SDSs

- *What dynamical properties are preserved as the update sequence changes?*
- How many SDS maps up to equivalence are obtainable by varying the update sequence?
- Can we characterize equivalence combinatorially, through properties of the base graph?

Key observation: We can associate an update sequence $\pi \in S_Y$ with an acyclic orientation, $O_Y^\pi \in \mathrm{Acyc}(Y)$, by a well-defined map

$$f_Y \colon S_Y \longrightarrow \mathrm{Acyc}(Y)\,, \qquad f_Y(\pi) = O_Y^\pi\,,$$

where π is a linear extension of O_Y^π.

Explicitly, if $\pi = \pi_1\pi_2\cdots\pi_n$ then $\{i,j\} \in \mathrm{e}[Y]$ is oriented (i,j) iff i appears before j in π.

- For any $\pi, \sigma \in S_Y$, define $\pi \sim_\alpha \sigma$ iff $f_Y(\pi) = f_Y(\sigma)$. This is an equivalence relation on S_Y.

Equivalence on acyclic orientations

▶ For $O, O' \in \mathrm{Acyc}(Y)$ define $O \sim_{\bar{\alpha}} O'$ if $O' = \gamma(O)$ for some $\gamma \in \mathrm{Aut}(Y)$.

A cyclic 1-shift (left) of a linear extension of O_Y corresponds to converting a source of O_Y into a sink.

▶ This *source-to-sink* operation (or a "click") puts an equivalence relation on $\mathrm{Acyc}(Y)$, denoted $\sim_\kappa$.

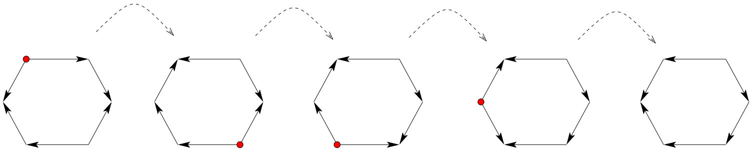


Figure: Source-to-sink operations

▶ $\mathrm{Aut}(Y)$-actions, with source-to-sink operations, together yield a coarser equivalence relation $\sim_{\bar{\kappa}}$.

Equivalence on update sequences

Let S_Y denote the set of permutation update sequences of $\mathrm{v}[Y]$.

- $\pi \sim_\alpha \sigma \quad \Longleftrightarrow \quad O_Y^\pi = O_Y^\sigma$.
- $\pi \sim_{\bar{\alpha}} \sigma \quad \Longleftrightarrow \quad O_Y^\pi \sim_{\bar{\alpha}} O_Y^\sigma$ (related by $\gamma \in \mathrm{Aut}(Y)$).
- $\pi \sim_\kappa \sigma \quad \Longleftrightarrow \quad O_Y^\pi \sim_\kappa O_Y^\sigma$ (related by source-to-sink moves).
- $\pi \sim_{\bar{\kappa}} \sigma \quad \Longleftrightarrow \quad O_Y^\pi \sim_{\bar{\kappa}} O_Y^\sigma$ (related by $\gamma \in \mathrm{Aut}(Y)$ & source-to-sink moves).

Theorem

Let $\mathfrak{F}_Y$ be a sequence of Aut(Y)*-invariant functions.*

- If $\pi \sim_\alpha \sigma$, then $[\mathfrak{F}_Y, \pi]$ and $[\mathfrak{F}_Y, \sigma]$ are *functionally equivalent.*
- If $\pi \sim_{\bar{\alpha}} \sigma$, then $[\mathfrak{F}_Y, \pi]$ and $[\mathfrak{F}_Y, \sigma]$ are *dynamically equivalent.*
- If $\pi \sim_\kappa \sigma$, then $[\mathfrak{F}_Y, \pi]$ and $[\mathfrak{F}_Y, \sigma]$ are *cycle equivalent.*
- If $\pi \sim_{\bar{\kappa}} \sigma$, then $[\mathfrak{F}_Y, \pi]$ and $[\mathfrak{F}_Y, \sigma]$ are *cycle equivalent.*

Enumeration problems

- $\alpha(Y) := \mathrm{Acyc}(Y) = T_Y(2,0)$ satisfies

$$\alpha(Y) = \alpha(Y/e) + \alpha(Y \setminus e) \qquad \text{for any edge } e$$

- $\bar{\alpha}(Y) := \mathrm{Acyc}(Y)/{\sim_{\bar{\alpha}}} = \dfrac{1}{\mathrm{Aut}(Y)} \displaystyle\sum_{\gamma \in \mathrm{Aut}(Y)} \alpha(\langle\gamma\rangle \setminus Y)$

- $\kappa(Y) := \mathrm{Acyc}(Y)/{\sim_\kappa} = T_Y(1,0)$ satisfies

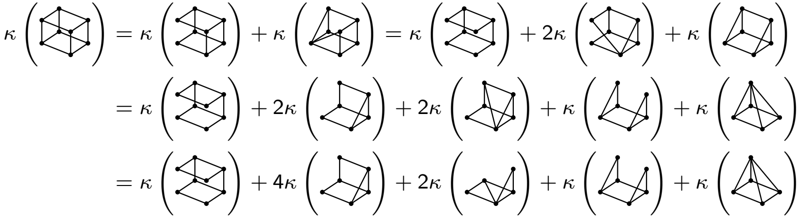
$$\kappa(Y) = \kappa(Y/e) + \kappa(Y \setminus e) \qquad \text{for any } \textit{cycle} \text{ edge } e$$

- $\bar{\kappa}(Y) := \mathrm{Acyc}(Y)/{\sim_{\bar{\kappa}}} = \dfrac{1}{\mathrm{Aut}(Y)} \displaystyle\sum_{\gamma \in \mathrm{Aut}(Y)} |\mathrm{Fix}(\gamma)|$

  But what is $|\mathrm{Fix}(\gamma)|$ *???*

An example

Let Q_2^3 be the binary 3-cube. A tedious calculation gives $\alpha(Y) = 1862$.



$$= 27 + 64 + 16 + 12 + 14 = 133.$$

- In summary, we have:

$$\alpha(Q_2^3) = 1862\,, \quad \bar{\alpha}(Q_2^3) = 54\,, \quad \kappa(Q_2^3) = 133\,, \quad \delta(Q_2^3) = 67\,, \quad \bar{\kappa}(Q_2^3) = \bar{\delta}(Q_2^3) = 8\ .$$

- If $Y = Q_2^3$, then for a fixed choice of functions $\mathfrak{F}_Y$, there *are at most* 8 possible cycle structures of the SDS map $[\mathfrak{F}_Y, \pi]$, up to isomorphism.

$\kappa(Y)$ for some special graph classes

Proposition ([6])

For $v \in \mathrm{v}[Y]$, let $|\mathrm{Acyc}_v(Y)|$ be the number of acyclic orientations of Y where v is the unique source. There is a bijection

$$\phi_v \colon \mathrm{Acyc}_v(Y) \longrightarrow \mathrm{Acyc}(Y)/\sim_\kappa \ .$$

Corollary

For any vertex v of Y the set $\mathrm{Acyc}_v(Y)$ is a transversal of $\mathrm{Acyc}(Y)/\sim_\kappa$.

- If Y is a tree, then $\kappa(Y) = 1$.
- If Y is an n-cycle, then $\kappa(Y) = n - 1$.
- If $Y \oplus v$ is the vertex join of Y, then $\kappa(Y \oplus v) = \alpha(Y)$.
- $\kappa(K_n) = (n-1)!$.

The ν-invariant

Let $P = v_1 v_2 \cdots v_k$ be a path in Y. Define $\nu_P(O_Y)$ to be the number of edges oriented (v_i, v_{i+1}), minus the number of edges oriented (v_{i+1}, v_i).

Easy fact: If P is a cycle, then $\nu_P(O_Y)$ is invariant under clicks.

Let $Y = \mathrm{Circ}_n$, and let P traverse Y once. The possible values for $\nu_P(\mathrm{Circ}_n)$ are $\pm(n-2), \pm(n-4), \pm(n-6), \dots$. Therefore, $\kappa(\mathrm{Circ}_n) \geq n-1$.

By the recurrence $\kappa(Y) = \kappa(Y/e) = \kappa(Y \setminus e)$, and with base case $\kappa(\mathrm{Tree}) = 1$, we get $\kappa_P(\mathrm{Circ}_n) = n-1$.

▶ Therefore, ν *is a complete invariant of* $\mathrm{Acyc}(\mathrm{Circ}_n)/{\sim_\kappa}$, i.e., if $Y = \mathrm{Circ}_n$,

$$\nu_P(O_Y) = \nu_P(O'_Y) \quad \Longleftrightarrow \quad O_Y \sim_\kappa O'_Y \, .$$

The ν-invariant (cont.)

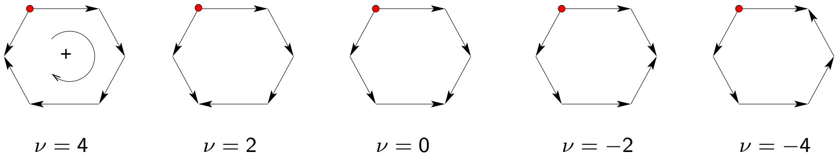


Figure: A transversal for $\mathrm{Acyc}(\mathrm{Circ}_6)/\sim_\kappa$.

In fact, taken over all cycles, ν is a *complete* invariant of $\mathrm{Acyc}(Y)/\sim_\kappa$:

Theorem (M–, Mortveit [7])

If $\nu_C(O_Y) = \nu_C(O'_Y)$ for every cycle C in Y, then $O_Y \sim_\kappa O'_Y$.

Coxeter groups

Definition

A *Coxeter group* is a group with presentation

$$\langle s_1, \ldots, s_n \mid s_i s_j^{m_{ij}} \rangle$$

where $m_{ij} > 1$ iff $i \neq j$.

It follows easily that $|s_i| = 2$, and $|s_i s_j| = |s_j s_i|$.

Think of a Coxeter group as a generalized reflection group (more on this later).

Recall, for any non-zero vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, the reflection of $\mathbf{v}$ across the hyperplane orthogonal to $\mathbf{w}$ is

$$\mathbf{v} - 2\frac{\langle \mathbf{v}, \mathbf{w} \rangle}{\langle \mathbf{w}, \mathbf{w} \rangle}\mathbf{w}\ .$$

Coxeter graphs and acyclic orientations

A *Coxeter system* is a triple (W, S, Γ) where W is a Coxeter group, S is the set of reflections that generate W, and Γ is the *Coxeter graph*:

$$\mathrm{v}[\Gamma] = S, \qquad \mathrm{e}[\Gamma] = \{\{s_i, s_j\} \mid m_{ij} \geq 3\} .$$

Additionally, each edge $\{s_i, s_j\}$ is labeled with m_{ij} (usually the label is omitted for $m_{ij} = 3$ because these are the most common).

Note: Edges correspond to non-commuting pairs of reflections.

- A *Coxeter element* is the product of the generators in any order.

- There is a bijection between the set of Coxeter elements $C := C(W, S, \Gamma)$ and $\mathrm{Acyc}(\Gamma)$ (see [11]).

Conjugacy of Coxeter elements

▶ Conjugating a Coxeter element by a simple reflection cyclically shifts the word, and corresponds to a *source-to-sink* operation (or "click"):

$$s_{\pi(1)}(s_{\pi(1)}s_{\pi(2)}\cdots s_{\pi(n)})s_{\pi(1)} = s_{\pi(2)}s_{\pi(3)}\cdots s_{\pi(n)}s_{\pi 1}\ .$$

Therefore, the equivalence relation $\sim_\kappa$ carries over to $C(W,S,\Gamma)$.

▶ Clearly, if $\mathbf{c} \sim_\kappa \mathbf{c}'$, then $\mathbf{c}$ and $\mathbf{c}'$ are conjugate in W.

▶ Therefore, $\kappa(\Gamma)$ is an upper bound on the number of conjugacy classes of Coxeter elements [6].

Open question: Is this bound sharp, i.e., does the converse of the statement above hold?

Conjugacy in simply-laced Coxeter groups

A Coxeter system is *simply-laced* if $m_{ij} \leq 3$.

Theorem (H. Eriksson, 1994 [2])

Let (W, S, Γ) be a simply-laced Coxeter system where $\Gamma = \mathrm{Circ}_n$ (i.e., $W = \tilde{A}_{n-1}$ is the affine Weyl group). Then two Coxeter elements $c, c' \in C(W, S, \Gamma)$ are conjugate if and only if $c \sim_\kappa c'$.

Theorem (J.-Y. Shi, 2001 [12])

Let (W, S, Γ) be a simply-laced Coxeter system where Γ is unicyclic. Then two Coxeter elements $c, c' \in C(W, S, \Gamma)$ are conjugate if and only if $c \sim_\kappa c'$.

Theorem (M–, Mortveit, 2008 [7])

Let (W, S, Γ) be a simply-laced Coxeter system. Then two Coxeter elements $c, c' \in C(W, S, \Gamma)$ are conjugate if and only if $c \sim_\kappa c'$.

Natural reflection representation

Define $a_{i,j} = \cos \frac{\pi}{m_{ij}}$.

The *natural reflection representation* of W is defined on the generators $s \in S$ by

$$s_i \longmapsto I_n - 2E_{i,i} + \sum_{j \,:\, m_{ij} \geq 3} a_{i,j} E_{i,j} \ .$$

Example.

$$s_i \quad \longmapsto \quad \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ \cdots & a_{i-1,i} & -1 & a_{i,i+i} & \cdots & & \\ & & & 1 & & & \\ & & & & \ddots & & \\ & & & & & & 1 \end{bmatrix}$$

Spectral classes

Identify $w \in W$ with the corresponding linear transformation in the natural reflection representation.

If w and w' are conjugate in W, then they have the same spectral class.

Question [12]: Given a Coxeter system (W, S, Γ), how many spectral classes do the Coxeter elements in $C(W, S, \Gamma)$ fall into?

Two κ-classes that have respective acyclic orientations O_Γ and O'_Γ such that $\varphi\colon O_\Gamma \longmapsto O'_\Gamma$ for some $\varphi \in \mathrm{Aut}(\Gamma)$ also have the same spectral class.

▶ Therefore, $\bar{\kappa}(\Gamma)$ is an upper bound for the number of spectal classes.

An example

Let $\Gamma = K_{2,3}$, with vertex set $\{1, 3, 5\} \sqcup \{2, 4\}$.

$\alpha(\Gamma) = 46$, $\kappa(\Gamma) = 7$, and $\bar{\kappa}(\Gamma) = 2$. There are 2 spectral classes (See Shi, 2001 [12]):

$$\{12345, 23451, 52341, 51234, 45123, 34512\}$$
$$\{12543, 25431, 32541, 31254, 43125, 54312\}$$
$$\{32145, 35214, 52143, 21435, 14352, 43521\}$$
$$\{14523, 45231, 34512, 31452, 23145, 52314\}$$
$$\{14325, 43251, 54321, 51432, 25143, 32514\}$$
$$\{34124, 41235, 54123, 35412, 23541, 12354\}$$
$$\{24351, 21354, 13524, 41352, 52431, 15243, 12435, 31245, 32451, 35241\}$$

Elements in the first six classes have characteristic polynomial $f(x) = x^5 - 3x^4 - 6x^3 - 6x^2 - 3x + 1$.

Elements in the last class have characteristic polynomial $f(x) = x^5 - x^4 - 8x^3 - 8x^2 - x + 1$.

An example (cont.)

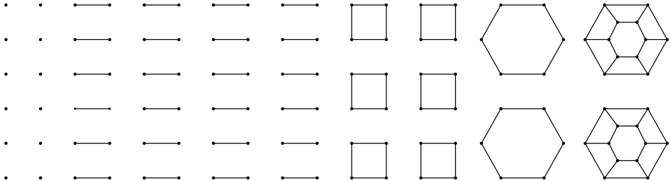


Figure: The update graph $U(K_{2,3})$: Connected components are in 1–1 correspondence with $\mathrm{Acyc}(K_{2,3})$.

Consider the mapping $(s_{\pi_i})_i \stackrel{\phi}{\longmapsto} (\pi_i \mod 2)_i$.

Non-adjacency in Γ coincides with parity, that is, if $\mathbf{c} = \mathbf{c}'$, then $\phi(\mathbf{c}) = \phi(\mathbf{c}')$.

- 12 size-1 components: 10101
- 24 size-2 components: 01011, 11010, 01101, 10110.
- 6 size-4 components: 10011, 11001.
- 2 size-6 components: 01110
- 2 size-12 components: 11100, 00111.

An example (cont.)

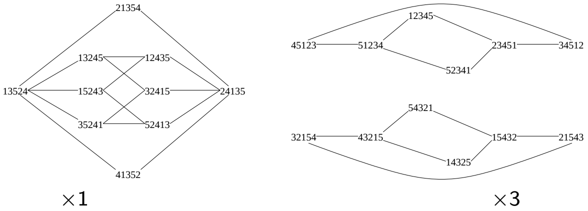


Figure: The graph $C(K_{2,3})$ contains the component on the left, and three isomorphic copies of the structure on the right (but with different vertex labels).

- Component at left: $\phi(\pi) \in \{01101, 11010, 10101, 01011, 10110\}$.
- Component at right: $\phi(\pi) \in \{11100, 11001, 10011, 00111, 01110\}$.

Quiver representations [8]

A *quiver* is a finite directed graph (loops and multiple edges allowed).

A quiver Q with a field K gives rise to a path algebra KQ.

There is a natural correspondence (categorial equivalence) between KQ-modules, and K-representations of Q.

▶ A path algebra is finite-dimensional if and only if the quiver is acyclic. Modules over finite-dimensional path algebras form a reflective subcategory.

▶ A *reflection functor* maps representations of a quiver Q to representations of a quiver Q', where Q' differs from Q by a source-to-sink operation.

▶ A composition of $n = |\mathrm{v}[Q]|$ distinct reflection functors is not the identity, but a *Coxeter functor*.

Node-firing games [3]

▶ In the *chip-firing game*, each vertex of a graph is given some number (possibly zero) of chips.

If vertex i has degree d_i, and at least d_i chips, then a legal move (or a "click") is a transfer of one chip to each neighbor.

A legal move is in a sense a generalization of a source-to-sink operation.

▶ In the *numbers game*, each vertex of a graph is assigned an integer value, and the edges are weighted according to the m_{ij} relations of the Coxeter group.

The legal sequences of moves in the numbers game are in 1–1 correspondence with the reduced words of the Coxeter group with that Coxeter graph.

Summary of SDS / Coxeter theory connections

		Coxeter groups	**Sequential dynamical systems**
Base graph	$\longleftrightarrow$	Coxeter graph Γ	Dependency graph Y
$\mathrm{Acyc}(\Gamma)$	$\longleftrightarrow$	Coxeter elements $c = s_{\pi(1)} s_{\pi(2)} \cdots s_{\pi(n)}$	SDS maps $[\mathfrak{F}_Y, \pi] = F_{\pi(n)} \circ \cdots \circ F_{\pi(2)} \circ F_{\pi(1)}$.
Clicks	$\longleftrightarrow$	Conjugacy classes of Coxeter elements	Cycle-equivalence classes of SDS maps
$\mathrm{Aut}(\Gamma)$ orbits	$\longleftrightarrow$	Spectral classes of Coxeter elements	Cycle-equivalence classes of SDS maps (finer)

Connections to quiver representations and chip firing

		Quiver representations	**Chip-firing game**
Base graph	$\longleftrightarrow$	Undirected quiver $\bar{Q}$	Underlying graph Γ
$\mathrm{Acyc}(\Gamma)$	$\longleftrightarrow$	Quiver Q of a finite-dimensional path-algebra KQ	Configurations, or states of the game
Clicks	$\longleftrightarrow$	Reflection functors	Legal moves
$\mathrm{Aut}(\Gamma)$	$\longleftrightarrow$	Vector space isomorphisms	Equivalent states

Summary of future research

Combinatorics

- Is there a nice closed-form or easily computable solution to $\bar{\kappa}(\Gamma)$?

Sequential dynamical systems

- Are $\bar{\kappa}(Y)$ and $\bar{\delta}(Y)$ sharp upper bounds for the number of SDS maps up to cycle equivalence?

Coxeter groups

- Prove that two Coxeter elements are conjugate iff they are κ-equivalent, for a non-simply-laced Coxeter system.
- Is $\bar{\kappa}(\Gamma)$ a sharp upper bound for the number of spectral classes of Coxeter elements of (W, S, Γ)? If not, for which graphs does it fail, and by how much?

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