Graph Dynamical Systems, Rank Functions, and Coxeter Groups

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Mathematics Colloquium
Clemson University
October 19, 2007
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   - Sequential dynamical systems
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2. Coxeter Groups
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3. Update Order Instability
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   - Summary of Stability
Consider the problem of counting the number of chambers $|C(\mathcal{H})|$ of a hyperplane arrangement $\mathcal{H}$ in $\mathbb{R}^n$.

**Example:**

![Figure: Cutting the sphere with hyperplanes](image)

The number of chambers depends not only on the number of hyperplanes, but also on the linear dependencies of the normal vectors. This is a problem handled by *matroids*.
Consider a hyperplane arrangement $\mathcal{H} = \{H_1, \ldots, H_k\}$, with corresponding normal vectors $\mathcal{V} = \{v_1, \ldots, v_k\}$.

If the normal vectors are linearly independent, then $|C(\mathcal{H})| = 2^k$.

If the hyperplanes (normal vectors) are in general position, $|C(\mathcal{H})| = 2 \sum_{i=0}^{n-1} \binom{k-i}{i}$.

Define the rank function of $\mathcal{V}$ by

$$r : \mathcal{P}(\mathcal{V}) \longrightarrow \mathbb{Z}, \quad r(\{v_{i_1}, \ldots, v_{i_k}\}) = \dim \langle v_{i_1}, \ldots, v_{i_k} \rangle.$$

$|C(\mathcal{H})|$ depends only on the rank function of $\mathcal{V}$. 
Definition

- A *sequential dynamical system* (SDS) is a triple consisting of:
  
  - A graph $Y$ with vertex set $\nu(Y) = \{1, 2, \ldots, n\}$.
  - For each vertex $i$ a state $y_i \in K$ (e.g. $F_2 = \{0, 1\}$) and a local function $F_i : K^n \rightarrow K^n$
    
    $$F_i(y = (y_1, y_2, \ldots, y_n)) = (y_1, \ldots, y_{i-1}, f_i(y[i]), y_{i+1}, \ldots, y_n).$$
    
    vertex function
  
  - A ordering $\pi = \pi_1 \pi_2 \cdots \pi_n \in S_Y$ of the vertex set.

- The SDS map generated by the triple $(Y, (F_i)_1^n, \pi)$ is
  
  $$[\mathcal{Y}, \pi] = F_{\pi_n} \circ F_{\pi_{n-1}} \circ \cdots \circ F_{\pi_1}.$$
Update graphs

Question: When does $[\mathcal{F}_Y, \pi] = [\mathcal{F}_Y, \sigma]$, for distinct update orders $\pi, \sigma \in S_Y$?

Definition. The update graph $U(Y)$ has vertex set $S_Y$. The edge $\{\pi, \sigma\}$ is present iff:

- $\pi$ and $\sigma$ different by exactly an adjacent transposition $(i, i + 1)$,
- $\{\pi_i, \pi_{i+1}\} \notin e[Y]$.

Example. Let Circ$_4$ be the circular graph on 4 vertices.

```
1234  2341  1243-----1423  1324-----1342
3412  4123  3241-----3421  3124-----3142
1432  2143  2134-----2314  2413-----2431
3214  4321  4132-----4312  4213-----4231
```

Figure: The update graph $U$(Circ$_4$).
Functional equivalence

Define an equivalence relation \( \sim_Y \) on \( S_Y \) by \( \pi \sim_Y \sigma \) if \( \pi \) and \( \sigma \) are on the same connected component of \( U(Y) \).

**Prop.** If \( \pi \sim_Y \sigma \), then \([\widehat{\delta}_Y, \pi] = [\widehat{\delta}_Y, \sigma]\).

\( \triangleright \) An ordering \( \pi \in S_Y \) induces an acyclic orientation of \( Y \), denoted \( O^\pi_Y \).

\( \triangleright \) There is a bijection between

\[
f_Y : S_Y/\sim_Y \longrightarrow \text{Acyc}(Y), \quad f_Y([\pi]_Y) = O^\pi_Y.
\]

Thus, \( \alpha(Y) = |\text{Acyc}(Y)| \) is an upper bound for the number of functionally distinct SDS maps \([\widehat{\delta}_Y, \pi]\) for a fixed choice of \( \widehat{\delta}_Y \). This bound is known to be sharp.

\( \triangleright \) The function \( \alpha \) satisfies the recurrence relation:

\[
\alpha(Y) = \alpha(Y_e') + \alpha(Y_e'') ,
\]

where \( Y_e' \) and \( Y_e'' \) are formed from \( Y \) by deleting and contracting edge \( e \).
Permutahedra

The \textit{n-permutahedron} $\Pi_n$ is the convex hull of all permutations of the points $(1, 2, \ldots, n) \in \mathbb{R}^n$. It is an $(n - 1)$-dimensional polytope.

The vertices and edges of $\Pi_n$ can be labeled as follows:

- Two vertices are adjacent if they differ by swapping two coordinates in adjacent position.
- An edge is labeled with a transposition $(x_i, x_j)$ of the values of the two entries that are swapped.

\textit{Note:} This labeling scheme does not agree with the geometric coordinates of the vertices!
Figure: Permutahedra, for $n = 3$ and $n = 4$. 
Constructing $U(Y)$ from $\Pi_n$

- $\Pi_n$ is the update graph of $E_n$.

- Each transposition $(i, j) \in S_n$ corresponds with a complete set of parallel edges of $\Pi_n$.

- The update graph $U(Y)$ can be constructed by “cutting” $\Pi_n$ with a hyperplane $H^n_{i,j}$ for every edge $\{i, j\} \in e[Y]$. 
An example

Figure: Hyperplanes cuts corresponding with the edges \{1, 2\}, \{2, 3\}, and \{1, 3\} in $Y < K_4$. 

(a) $Y$

(b) Constructing $U(Y)$
The rank function of a graph

**Definition.** The *rank function* of a graph \( Y \) is the function

\[
r_Y : \mathcal{P}(Y) \rightarrow \mathbb{Z} , \quad r_Y(Z) = r(\{v_{i,j}^n | \{i,j\} \in e[Z]\}) ,
\]

where \( v_{i,j}^n \) is the normal vector of the hyperplane \( H_{i,j}^n \) from the constructing of \( U(Y) \).

**Definition.** Let \( f_1, \ldots, f_n \) be a basis for the dual space of \( \mathbb{R}^n \). For a graph \( G = (V,E) \), let \( \mathcal{H}(G) \) be the arrangement defined by

\[
\mathcal{H}(G) = \{ \ker(f_i - f_j) | \{i,j\} \in E \} .
\]

\( \mathcal{H}(G) \) is called the *graphic arrangement* of \( G \).

**Prop.** If \( Z < Y \), then \( r_Y(Z) \) is the number of edges in a spanning forest of \( Z \), i.e.,

\[
r_Y(Z) = |v[Z]| - n(Z) .
\]
A Coxeter group is a group with presentation

\[ C = \langle r_1, \ldots, r_n \mid r_i^2, (r_i r_j)^{m_{ij}} \ (i \neq j) \rangle. \]

A Coxeter element is the product of the generators in any order.

Every Coxeter group \( C \) has a Coxeter graph \( \Gamma(C) \) with vertex set \( \{r_1, \ldots, r_n\} \) and edges \( \{r_i, r_j\} \) labeled with \( m_{ij} \) iff \( m_{ij} \geq 3 \).

Prop. There is a bijection between the Coxeter elements of \( C \) and the acyclic orientations of \( \Gamma(C) \).
Source-to-sink moves

- Cyclically shifting a Coxeter element corresponds with conjugation:
  \[ r_{\pi_1} \left( r_{\pi_1} r_{\pi_2} \cdots r_{\pi_n} \right) r_{\pi_1} = r_{\pi_2} r_{\pi_3} \cdots r_{\pi_n} r_{\pi_1} . \]

- This corresponds to a *click* of vertex \( r_{\pi_1} \): changing it from a source to a sink in the acyclic orientation of the Coxeter graph \( Y = \Gamma(C) \).

- This puts an equivalence relation \( \sim_{\kappa} \) on \( \text{Acyc}(Y) \), (and thus on \( S_Y / \sim_Y \)).

- The function \( \kappa(Y) = | \text{Acyc}(Y) / \sim_{\kappa} | \) is an upper bound for the number of conjugacy classes of Coxeter elements. (Is it sharp???)
Cycle equivalence

**Definition** Two finite dynamical systems $\phi, \psi : K^n \to K^n$ are *cycle equivalent* if there exists a bijection $h : \text{Per}(\phi) \to \text{Per}(\psi)$ such that

$$\psi|_{\text{Per}(\psi)} \circ h = h \circ \phi|_{\text{Per}(\phi)}.$$ 

Let $\sigma, \tau \in S_n$ be

$$\sigma = (n, n-1, \ldots, 2, 1), \quad \tau = (1, n)(2, n-1) \cdots ([n/2], [n/2] + 1),$$

and let $C_n$ and $D_n$ be the groups

$$C_n = \langle \sigma \rangle, \quad D_n = \langle \sigma, \tau \rangle.$$ 

These groups act on update orders $\pi = \pi_1 \pi_2 \cdots \pi_n$ by *shift* and *reflection* as follows:

$$\sigma(\pi) := \sigma \cdot \pi = \pi_2 \pi_3 \cdots \pi_n \pi_1, \quad \tau(\pi) := \tau \cdot \pi = \pi_n \pi_{n-1} \cdots \pi_2 \pi_1.$$
Cycle equivalence of SDSs

**Theorem**

For any $\pi \in S_Y$, the SDS maps $[\mathfrak{S}_Y, \pi]$ and $[\mathfrak{S}_Y, \sigma(\pi)]$ are cycle equivalent. Moreover, if $K = \mathbb{F}_2$, then these are cycle equivalent to $[\mathfrak{S}_Y, \tau(\pi)]$ as well.

**Example.** Define the function $\text{nor} : \mathbb{F}_2^k \rightarrow \mathbb{F}_2$ by $\text{nor}(x) = \prod_{i=1}^{k}(1 + x_i)$.

![Diagram](image)

Figure: Phase spaces of an SDS with different update orders.
The function $\kappa(Y)$

- The function $\kappa(Y) := |\text{Acyc}(Y)/\sim_\kappa|$ is thus an upper bound for:
  - The number of conjugacy classes of Coxeter elements of a Coxeter group with Coxeter graph $Y$.
  - The number of SDS maps $[\mathfrak{F}_Y, \pi]$ up to cycle-equivalence for a fixed choice of $\mathfrak{F}_Y$.

Theorem

*If $e$ is a bridge edge of $Y$ linking components $Y_1$ and $Y_2$, then*

$$\kappa(Y) = \kappa(Y_1) \kappa(Y_2).$$

*If $e$ is a non-bridge edge of $Y$, then*

$$\kappa(Y) = \kappa(Y_e') + \kappa(Y_e'').$$
The case $K = \mathbb{F}_2$ and $\delta(Y)$

- The function $\delta(Y) := |\text{Acyc}(Y)/\sim_\delta|$ is an upper bound for the number of SDS maps $[\mathfrak{F}_Y, \pi]$ up to cycle-equivalence for a fixed choice of $\mathfrak{F}_Y$ when $K = \mathbb{F}_2$.

**Theorem**

*If $Y$ is a connected undirected graph, then*

$$\delta(Y) = \begin{cases} \frac{1}{2}(\kappa(Y) + 1) & \text{if } Y \text{ is bipartite}, \\ \frac{1}{2}\kappa(Y) & \text{if } Y \text{ is not bipartite}. \end{cases}$$
An example

Let $Q^3_2$ be the binary 3-cube. A tedious calculation gives $\alpha(Y) = 1862$.

$$\kappa \left( \begin{array}{c}
\text{cube} \\
\text{cube}
\end{array} \right) = \kappa \left( \begin{array}{c}
\text{cube} \\
\text{cube}
\end{array} \right) + \kappa \left( \begin{array}{c}
\text{cube} \\
\text{cube}
\end{array} \right) = \kappa \left( \begin{array}{c}
\text{cube} \\
\text{cube}
\end{array} \right) + 2\kappa \left( \begin{array}{c}
\text{cube} \\
\text{cube}
\end{array} \right) + \kappa \left( \begin{array}{c}
\text{cube} \\
\text{cube}
\end{array} \right)$$

$$= \kappa \left( \begin{array}{c}
\text{cube} \\
\text{cube}
\end{array} \right) + 2\kappa \left( \begin{array}{c}
\text{cube} \\
\text{cube}
\end{array} \right) + 2\kappa \left( \begin{array}{c}
\text{cube} \\
\text{cube}
\end{array} \right) + \kappa \left( \begin{array}{c}
\text{cube} \\
\text{cube}
\end{array} \right) + \kappa \left( \begin{array}{c}
\text{cube} \\
\text{cube}
\end{array} \right)$$

$$= \kappa \left( \begin{array}{c}
\text{cube} \\
\text{cube}
\end{array} \right) + 4\kappa \left( \begin{array}{c}
\text{cube} \\
\text{cube}
\end{array} \right) + 2\kappa \left( \begin{array}{c}
\text{cube} \\
\text{cube}
\end{array} \right) + \kappa \left( \begin{array}{c}
\text{cube} \\
\text{cube}
\end{array} \right) + \kappa \left( \begin{array}{c}
\text{cube} \\
\text{cube}
\end{array} \right)$$

$$= 27 + 64 + 16 + 12 + 14 = 133$$

Thus, $\kappa(Q^3_2) = 133$, and $\delta(Q^3_2) = 67$. But...many of these equivalence classes are related by some automorphism of $Q^3_2$. By acting on $Q^3_2$ by $\text{Aut}(Q^3_2)$ an applying Burnside’s Lemma, we get 8 distinct orbits. Therefore:

If $Y = Q^3_2$, then for a fixed choice of functions $\mathfrak{F}_Y$, there are at most 8 possible cycle structures of the SDS map $[\mathfrak{F}_Y, \pi]$, up to isomorphism.
Cycle equivalence as a measure of stability

**Prop.** If $Y$ is a tree, then $\kappa(Y) = 1$.

**Corollary.** If $Y$ is a tree, then for a fixed choice of functions $\mathcal{F}_Y$, all SDS maps $[\mathcal{F}_Y, \pi]$ are cycle equivalent.

- Adding more edges to $Y$ can only increase $\kappa(Y)$, and thus the number of possible cycle structures of SDS maps.

**Conclusion.** In general, adding edges to the dependency graph of an SDS causes the dynamics to become less stable with respect to update order.
Reachable attractor basins as a measure of stability

Consider a sequential dynamical system over a graph $Y$, with 2-threshold functions.

- If $Y$ is a tree, then there are states that can reach $2^n - n = O(2^n)$ fixed points by varying the update order.

- If $Y = K_n$, then any given state can reach at most $n + 1$ fixed points by varying the update order.

- In a recent paper, we have extended this to $G_{n,p}$ for various values of $p$.

**Conclusion.** In general, adding edges to the dependency graph of an SDS causes the dynamics to become *more* stable with respect to update order.
Word-independence as a measure of stability

▶ For some SDSs, the set of periodic points is independent of the update order. Such systems are said to be word-independent.

▶ Fixed point systems (such as threshold systems) are word-independent.

▶ Being word-independent seems to be more of a property of the functions, rather than the graph.

**Conclusion.** In general, adding edges to the dependency graph of an SDS has *very little* effect on the stability of the dynamical system.
What is going on???

We are discussing *different notions* of update order instability!

By update order stability, one can measure:

- How many different possible cycles structures (long term behaviors) are there... 
- How many different attractor basins can be reached from a particular state ... 
- Which states can arise as fixed points... 

...as the update order is perturbed.

*Moral:* Be *careful* when making general statements about the update order stability of a dynamical system!

*Question:* These ideas have only been studied independently. Is there a way to tie them together to paint a clearer picture?
Role of the graph structure in complex systems

When studying complex systems, it is natural to ask the following question:

**Question 1.** *What role does the structure of the dependency graph play in the dynamics of the a system defined over it?*

A good way to approach this is to ask another question:

**Question 2.** *Does there exist a good measure, or classification of graphs, useful for people studying graph dynamical systems?*

*Key idea:* Such a measure should be able to capture the cycle structure of the graph.

Measures such as the *degree distribution* or *clustering coefficient* can’t detect the presence of large cycles.
So... Why do we care about such a question?

(a) It’s an interesting problem
(b) It’s a novel idea
(c) There are real applications to modeling of biological, epidemiology, and social networks
(d) The NSF cares (and they’re rich)

Answer: All of the above!

Yes, the NSF has a passing interest, to the tune of $752 million over the next 5 years!

The objective of their Cyber-enabled Discovery and Innovation (CDI) program: Broaden the Nation’s capability for innovation by developing a new generation of computationally based discovery concepts and tools to deal with complex, data-rich, and interacting systems.
Approach: Edge shattering

- Recall that the rank function $r_Y$ captures the cycle structure of a graph, and is determined by how many components remain upon removal of a certain subset of edges.

- For a graph with $m$ edges, the rank function is of size $\Theta(2^m)$, thus it is uncomputable for most graphs.

- However, we can extract useful edge shattering properties from the following functions $[0,1] \rightarrow \mathbb{N}$:
  - $\mu_Y(k)$: average number of components when $k\%$ of edges are removed from $Y$.
  - $M_Y(k)$: maximum number of components when $k\%$ of edges are removed from $Y$.
  - $\lambda_Y(k)$: average size of largest component when $k\%$ of edges are removed from $Y$.
  - $\sigma_Y(k)$: average size of a component when $k\%$ of edges are removed from $Y$. 
Questions

- How well do these edge shattering functions distinguish commonly studied classes of graphs? For example:
  - Classical random graphs, such as $G_{n,p}$ and $G_{n,M}$.
  - Small-world networks (Watts & Strogatz, 1998)
  - Scale-free networks (Barabási and Albert, 1999)
  - Real-world biological, epidemiological, and social networks.

- Can one reconstruct a network of a particular size that satisfies certain edge shattering properties (hard!).
Acknowledgments

Joint work with: V. S. Anil Kumar, Reinhard Laubenbacher, Jon McCammond, Henning Mortveit.

Special thanks:

- Network Dynamics and Simulation Science Laboratory (NDSSL), at the Virginia Bioinformatics Institute,
- Los Alamos National Laboratory,
- The Fields Institute,
- All of you at Clemson University.

SDS course web page with link to papers:

Web: [http://www.math.vt.edu/people/hmortvei/class_home/4984_15748.html](http://www.math.vt.edu/people/hmortvei/class_home/4984_15748.html)

NDSSL:

Web: [http://ndssl.vbi.vt.edu](http://ndssl.vbi.vt.edu)