Visual Algebra

Lecture 3.5: Subgroup diagrams

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The conjugacy class of a subgroup

Proposition

Conjugation is an equivalence relation on the set of subgroups of G.

Proof

We need to show that conjugacy is reflexive, symmetric, and transitive.

- **Reflexive**: $eHe^{-1} = H$. ✓
- **Symmetric**: Suppose H is conjugate to K, by $aHa^{-1} = K$. Then K is conjugate to H:

$$a^{-1}Ka = a^{-1}(aHa^{-1})a = H. \quad \checkmark$$

- **Transitive**: Suppose $aHa^{-1} = K$ and $bKb^{-1} = L$. Then H is conjugate to L:

$$(ba)H(ba)^{-1} = b(aHa^{-1})b^{-1} = bKb^{-1} = L. \quad \checkmark$$

Definition

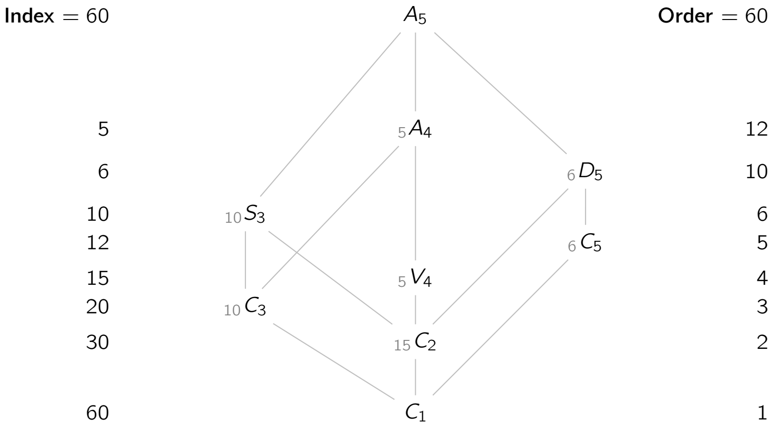
The set of all subgroups conjugate to H is its conjugacy class, denoted

$$\text{cl}_G(H) = \{gHg^{-1} \mid g \in G\}.$$

"Reducing" subgroup lattices

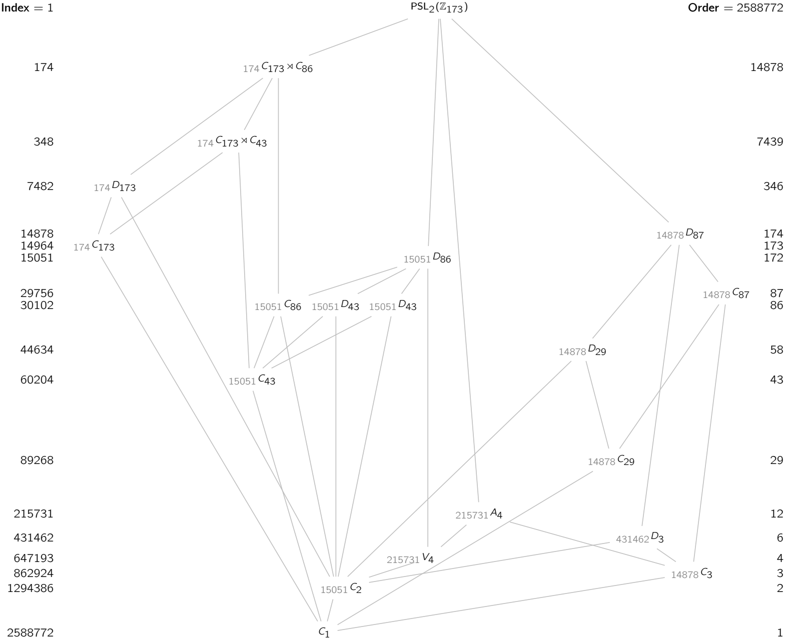
Sometimes it is convenient to collapse conjugacy classes into single nodes in the lattice.

Left-subscripts denote size. We call this a subgroup diagram. It need not be a lattice.



By inspection, we see this group is simple (its only normal subgroups are G and $\{e\}$).

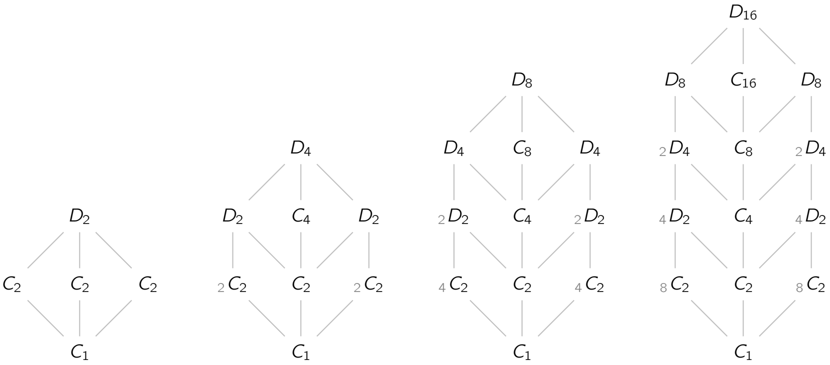
The subgroup diagram of a giant group



Subgroup diagrams of dihedral groups

Subgroup diagrams can reveal patterns that are hidden in the subgroup lattices.

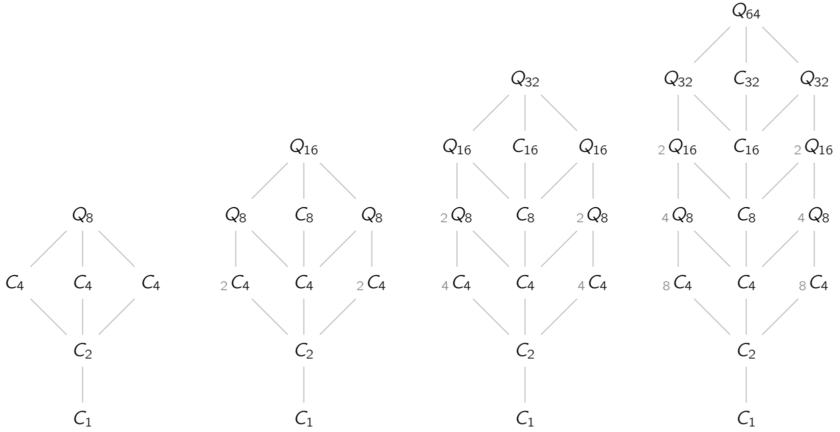
Here are the subgroup diagrams of the dihedral groups. (Note that $D_2 \cong V_4$.)



Subgroup diagrams of quaternion groups

Subgroup diagrams can reveal patterns that are hidden in the subgroup lattices.

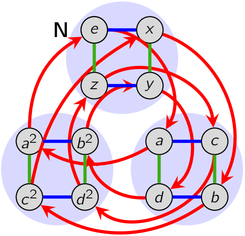
Here are the subgroup diagrams of the generalized quaternion groups. What do you notice?

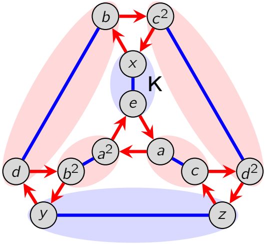


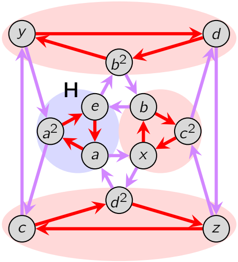
Revisiting A_4

The normalizer of each subgroup consists of the elements in the blue left cosets.

Here, take $a = (123)$, $x = (12)(34)$, $z = (13)(24)$, and $b = (234)$.







(124)	(234)	(143)	(132)
(123)	(243)	(142)	(134)
e	(12)(34)	(13)(24)	(14)(23)

$[A_4 : N_{A_4}(N)] = 1$

"normal"

(124)	(234)	(143)	(132)
(123)	(243)	(142)	(134)
e	(12)(34)	(13)(24)	(14)(23)

$[A_4 : N_{A_4}(K)] = 3$

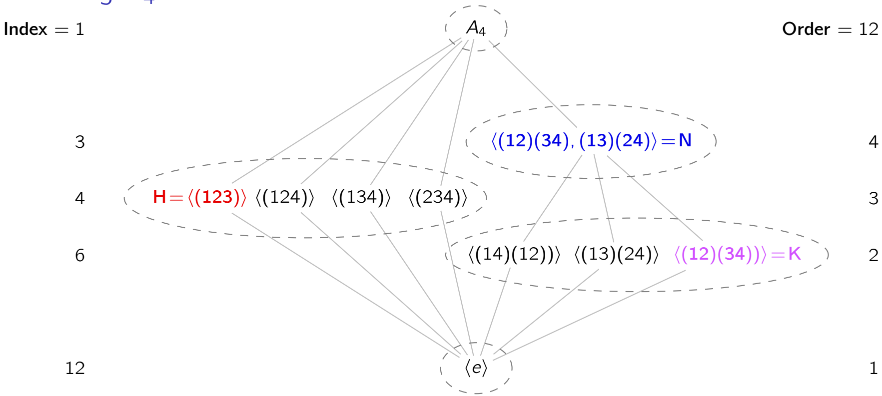
"moderately unnormal"

(14)(23)	(142)	(143)
(13)(24)	(243)	(124)
(12)(34)	(134)	(234)
e	(123)	(132)

$[A_4 : N_{A_4}(H)] = 4$

"fully unnormal"

Revisiting A_4



Observations

- A subgroup is normal if its conjugacy class has size 1.
- The size of a conjugacy class tells us *how close to being normal* a subgroup is.
- For our "three favorite subgroups of A_4":

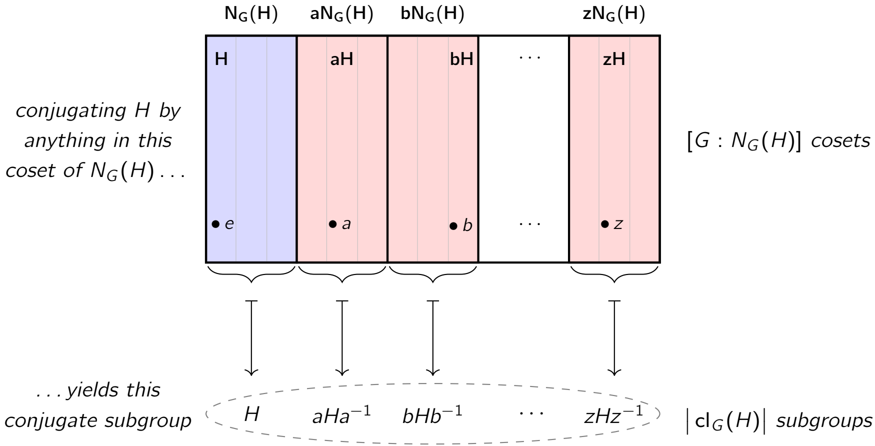
$$\left|\mathrm{cl}_{A_4}(N)\right| = 1 = \frac{1}{\mathrm{Deg}^{\triangleleft}_{A_4}(N)}, \quad \left|\mathrm{cl}_{A_4}(K)\right| = 3 = \frac{1}{\mathrm{Deg}^{\triangleleft}_{A_4}(K)}, \quad \left|\mathrm{cl}_{A_4}(H)\right| = 4 = \frac{1}{\mathrm{Deg}^{\triangleleft}_{A_4}(H)}.$$

The number of conjugate subgroups

Theorem

For any subgroup $H \leq G$, the **size of its conjugacy class** is the **index of its normalizer**:

$$\left|\text{cl}_G(H)\right| = [G : N_G(H)].$$



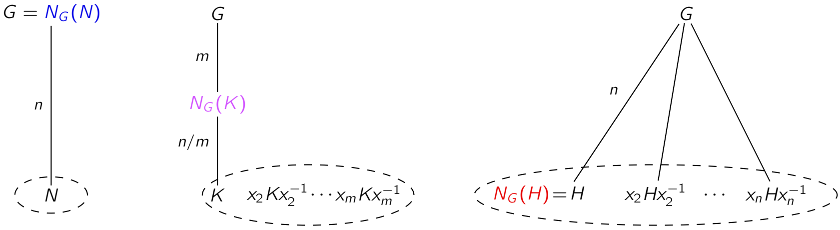
The number of conjugate subgroups

Theorem

Let $H \leq G$ with $[G : H] = n < \infty$. Then

$$\left|\mathrm{cl}_G(H)\right| = [G : N_G(H)] = \frac{\#\text{ elts } x \in G}{\#\text{ elts } x \in G \text{ for which } xH = Hx} = \frac{1}{\mathrm{Deg}_G^{\triangleleft}(H)}.$$

That is, H has exactly $[G : N_G(H)]$ conjugate subgroups.



normal — $\left|\mathrm{cl}_G(N)\right| = 1$

moderately unnormal — $1 < \left|\mathrm{cl}_G(K)\right| < [G : K]$

fully unnormal — $\left|\mathrm{cl}_G(H)\right| = [G : H]$; as large as possible

The “fan” of conjugate subgroups

The set of conjugate subgroups to $H \leq G$ “*looks like a fan in the lattice*”.

However not all such “fans” comprise a single conjugacy class.

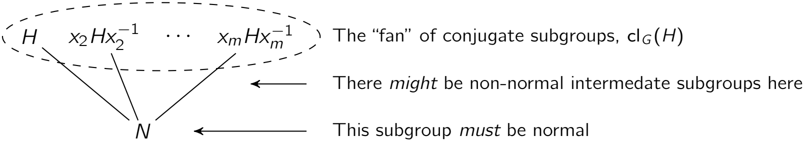
Big idea

- a “wide fan” means the subgroup is “very unnormal”, and has a smaller normalizer.
- a “narrow fan” means the subgroup is “closer to normal”, and has a larger normalizer.

The following says that “*the base of a fan is always normal*.”

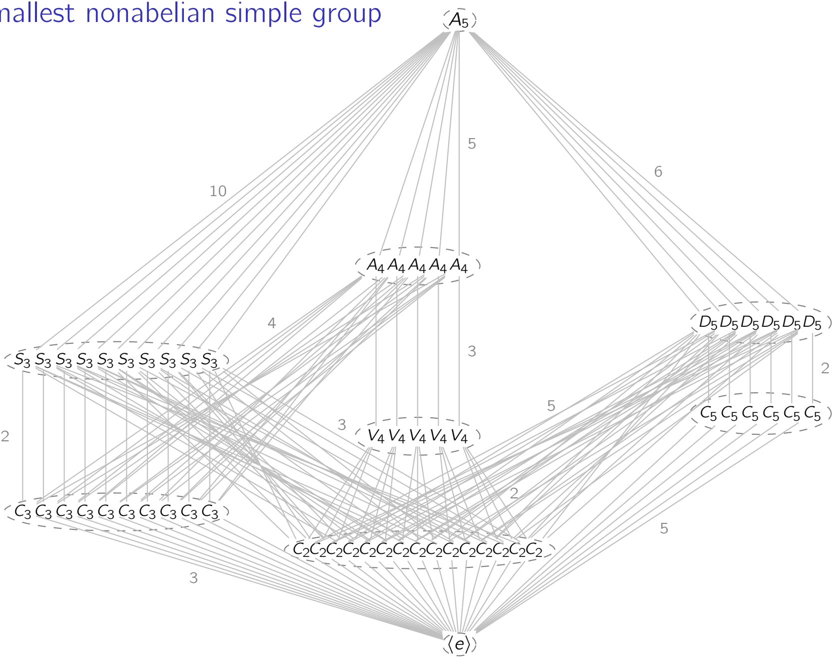
Proposition (exercise)

For any $H \leq G$, the intersection of all conjugates is normal: $N := \bigcap_{x \in G} xHx^{-1} \trianglelefteq G$.

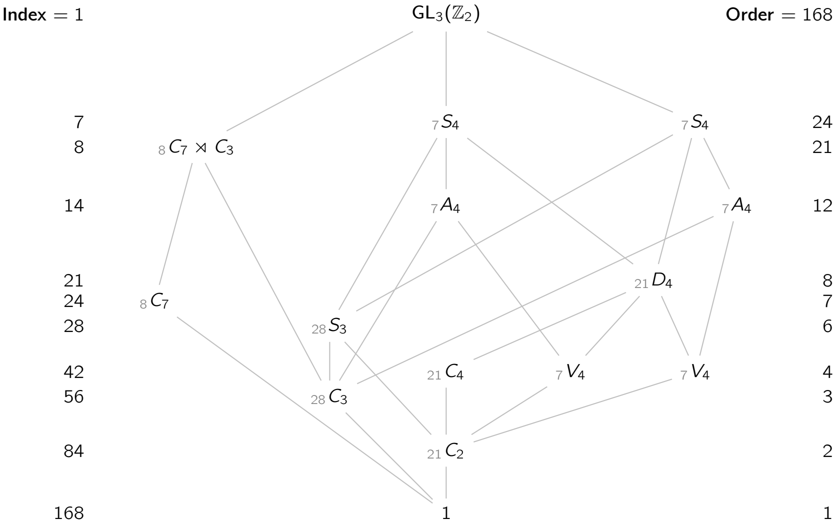


This places strong restrictions on the lattice structure of any simple group!

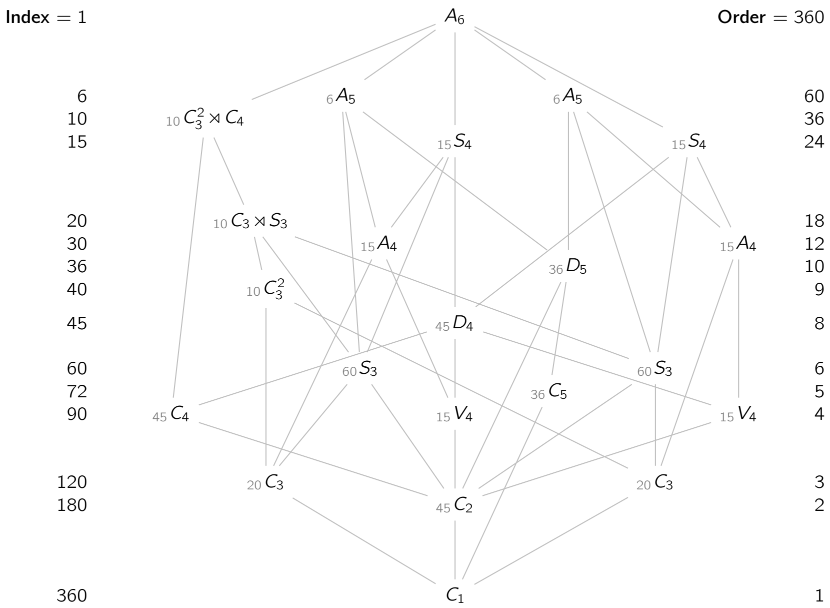
The smallest nonabelian simple group



The second smallest nonabelian simple group



The third smallest nonabelian simple group



The 71$^{\text{st}}$ smallest nonabelian simple group