# Visual Algebra

## Lecture 4.5: Quotients of quotient groups

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# The isomorphism theorems

Thus far, we have seen the first two of the four basic theorems about homomorphisms and their structure.

These are commonly called "The isomorphism theorems."

- Fundamental homomorphism theorem: "*All homomorphic images are quotients*"
- Correspondence theorem: Characterizes "*subgroups of quotients*"
- Fraction theorem: Characterizes "*quotients of quotients*"
- Diamond theorem: "*Duality of subquotients*."

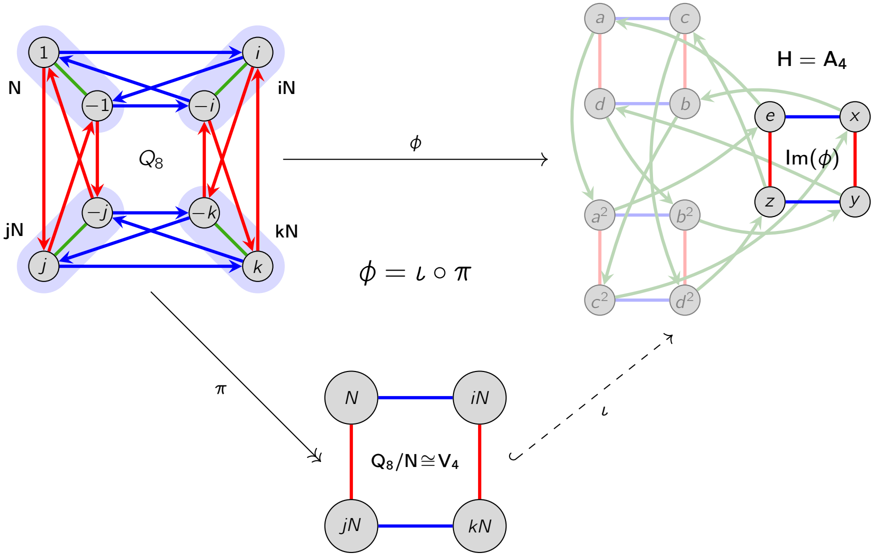
These all have analogues for other algebraic structures, e.g., rings, vector spaces, modules, Lie algebras.

All of these theorems can look messy and unmotivated algebraically.

However, they all have beautiful visual interpretations, especially involving subgroup lattices.

In this lecture, we'll study the fraction theorem.

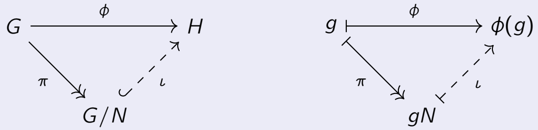
# Recall: a generalization of the FHT

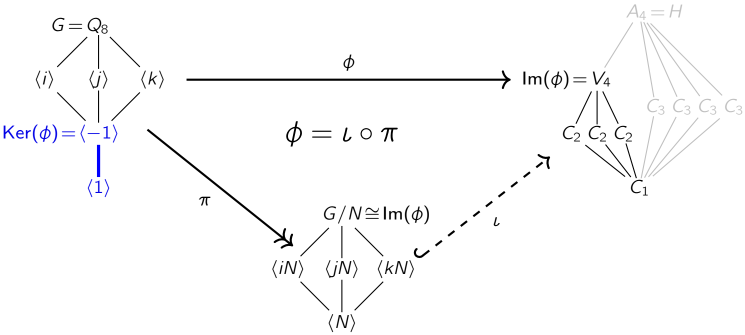


# A generalization of the FHT

## Theorem (exercise)

Every homomorphism $\phi\colon G \to H$ can be factored as a quotient and embedding:



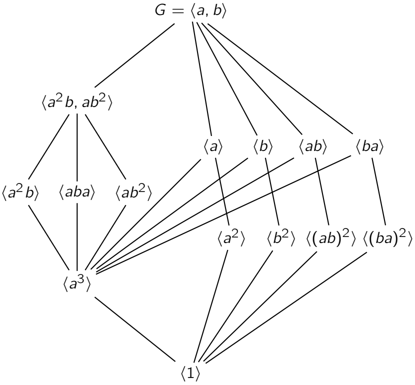



# The “subgroup” and “quotient” operations commute

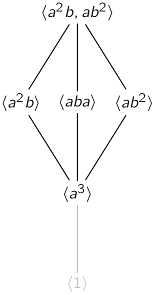
## Key idea

The quotient of a subgroup is just the subgroup of the quotient.

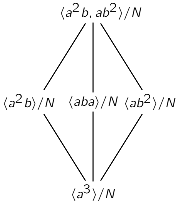
**Example**: Consider the group $G = \mathrm{SL}_2(\mathbb{Z}_3)$.




subgroup $\mathbf{H \cong Q_8}$




$\mathbf{H/N \cong V_4}$




“*quotient of the subgroup*”

# The "subgroup" and "quotient" operations commute

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The quotient of a subgroup is just the subgroup of the quotient.

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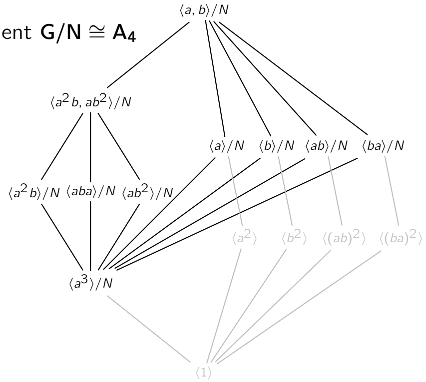
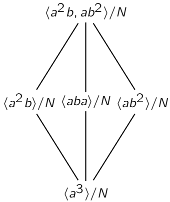



$\mathbf{V_4 \cong H/N \leq G/N}$




"*subgroup of the quotient*"

# The fraction theorem: quotients of quotients

The correspondence theorem characterizes the subgroup structure of the quotient $G/N$:

*Every subgroup of $G/N$ is of the form $H/N$, where $N \leq H \leq G$.*

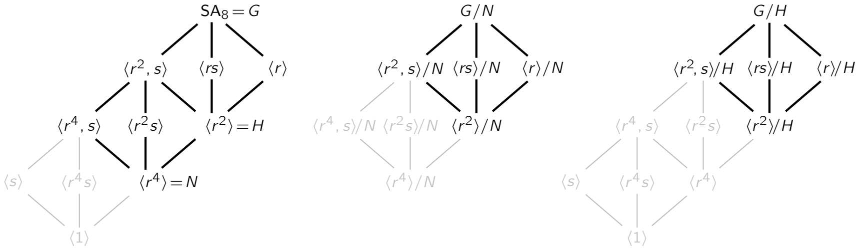
Moreover, if $H \trianglelefteq G$, then $H/N \trianglelefteq G/N$. In this case, we can ask:

*What is the quotient group $(G/N)/(H/N)$ isomorphic to?*

## Fraction theorem

Given a chain $N \leq H \leq G$ of normal subgroups of $G$,

$$(G/N)/(H/N) \cong G/H.$$



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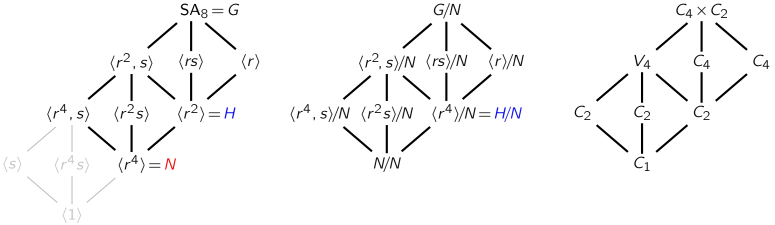
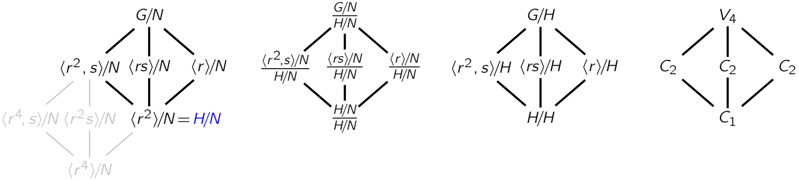
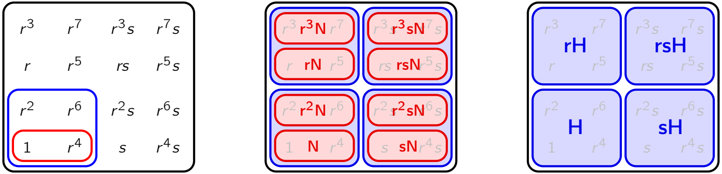


# The fraction theorem: quotients of quotients

Let's continue our example of the semiabelian group $G = \mathrm{SA}_8 = \langle r, s \rangle$.



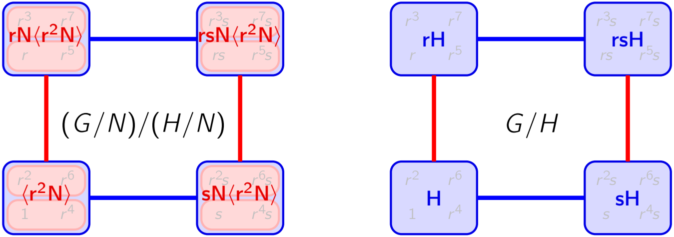

$N \leq H \leq G$

$G/N = \langle rN, sN \rangle \cong C_4 \times C_2$

$H/N = \langle r^2N \rangle = \{N, r^2N\} \cong C_2$

$G/H = \langle rH, sH \rangle \cong V_4$

$(G/N)/(H/N) \cong G/H$



# The fraction theorem: quotients of quotients

## Fraction theorem

Given a chain $N \leq H \leq G$ of normal subgroups of $G$,

$$(G/N)/(H/N) \cong G/H.$$

## Proof

This is tailor-made for the FHT. Define the map

$$\phi\colon G/N \longrightarrow G/H, \qquad \phi\colon gN \longmapsto gH.$$

- *Show $\phi$ is well-defined*: Suppose $g_1N = g_2N$. Then $g_1 = g_2n$ for some $n \in N$. But $n \in H$ because $N \leq H$. Thus, $g_1H = g_2H$, i.e., $\phi(g_1N) = \phi(g_2N)$. ✓
- *$\phi$ is clearly onto and a homomorphism*. ✓
- *Apply the FHT*:

$$\begin{aligned} \mathrm{Ker}(\phi) &= \{gN \in G/N \mid \phi(gN) = H\} \\ &= \{gN \in G/N \mid gH = H\} \\ &= \{gN \in G/N \mid g \in H\} = H/N \end{aligned}$$

By the FHT, $(G/N)/\mathrm{Ker}(\phi) = (G/N)/(H/N) \cong \mathrm{Im}(\phi) = G/H$. □

# The fraction theorem: quotients of quotients

For another visualization, consider $G = \mathbb{Z}_6 \times \mathbb{Z}_4$ and write elements as strings.

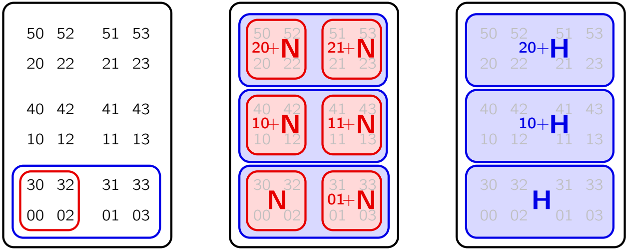
Consider the subgroups $N = \langle 30, 02 \rangle \cong V_4$ and $H = \langle 30, 01 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_4$.

Notice that $N \leq H \leq G$, and $H = N \cup (01{+}N)$, and

$$G/N = \big\{N,\ 01{+}N,\ 10{+}N,\ 11{+}N,\ 20{+}N,\ 21{+}N\big\}, \qquad H/N = \{N,\ 01{+}N\}$$

$$G/H = \Big\{N \cup (01{+}N),\ (10{+}N) \cup (11{+}N),\ (20{+}N) \cup (21{+}N)\Big\}$$

$$(G/N)/(H/N) = \Big\{\{N,\ 01{+}N\},\ \{10{+}N,\ 11{+}N\},\ \{20{+}N,\ 21{+}N\}\Big\}.$$




$N \leq H \leq G$

$G/N$ consists of 6 cosets
$H/N = \{N,\ 01{+}N\}$

$G/H$ consists of 3 cosets
$(G/N)/(H/N) \cong G/H$