Visual Algebra

Lecture 5.8: Simply transitive actions

Dr. Matthew Macauley

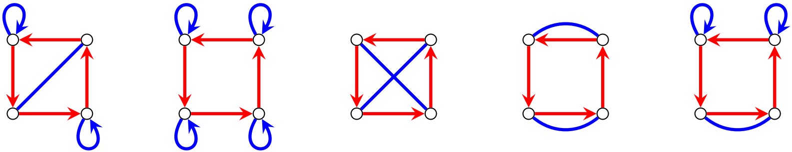
School of Mathematical & Statistical Sciences
Clemson University
South Carolina, USA
http://www.math.clemson.edu/~macaule/

Classification of G-sets

Natural question (answered last time)

Given a group G, what are its possible (connected) G-sets?

For example, which of the following can arise as an orbit of an action by $G = D_4$?



Definition

An action $\phi\colon G \to \mathrm{Perm}(S)$ is

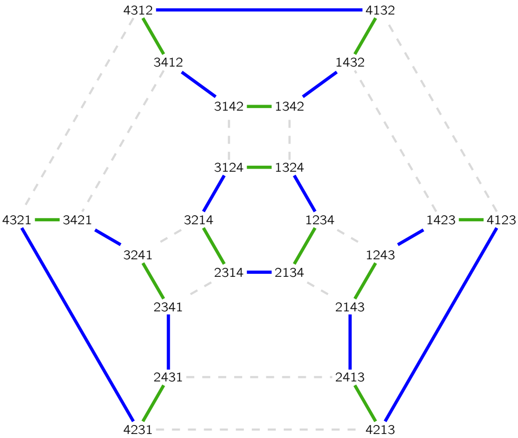
- transitive if it has only one orbit: ("*graph is connected*")
- free if $\mathrm{stab}(s) = \langle e \rangle$ for all $s \in S$. ("*uncollapsed – no nontrivial loops*")
- faithful if $\mathrm{Ker}(\phi) = \langle e \rangle$.

In this language our question becomes: "*classify all transitive G-actions*" (or G-sets).

An example of a free action that is not transitive

The group $S_3 = \langle (12), (23) \rangle$ acts on permutations **1234**, via $\phi \colon S_3 \to \mathsf{Perm}(S)$, where

- $\phi((12))$ = the permutation that swaps the 1st and 2nd coordinates
- $\phi((23))$ = the permutation that swaps the 2nd and 3rd coordinates

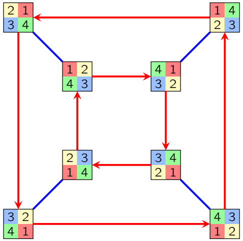


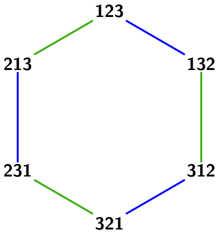
Simply transitive actions

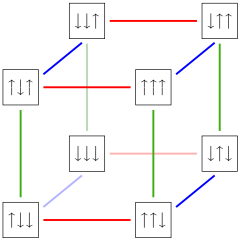
Definition

An action $\phi \colon G \to \mathsf{Perm}(S)$ is simply transitive if it is transitive and free.

Here are some simply transitive actions that we have seen.







Proposition

Every simply transitive G-action is isomorphic to G acting on itself by multiplication.

This just says that *simply transitive G-sets are groups*!

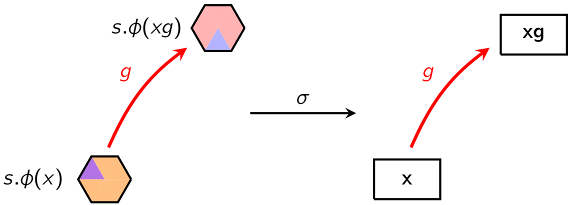
Simply transitive actions

Proposition

Every simply transitive G-action is isomorphic to G acting on itself by multiplication.

Proof sketch. Let $\iota\colon G \to G$ be the identity, fix our "home state" $s \in S$, and define

$$\sigma\colon S \longrightarrow G, \qquad \sigma\colon s.\phi(x) \longmapsto x$$



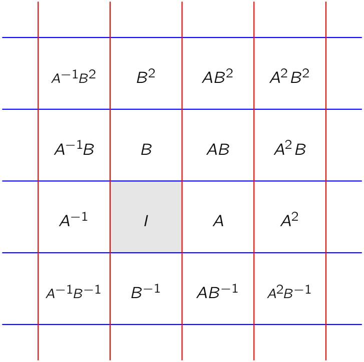
Show that σ is a well-defined bijection, and then the proof follows because:

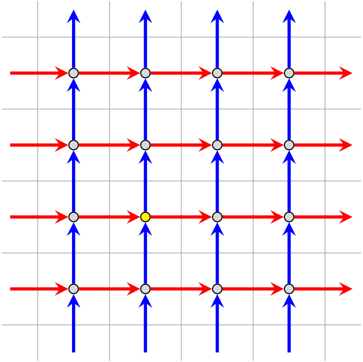
$$\begin{tikzcd} S \arrow[r, "\phi(g)"] \arrow[d, "\sigma"'] & S \arrow[d, "\sigma"] \\ G \arrow[r, "\phi(g)"'] & G \end{tikzcd} \qquad \begin{tikzcd} s.\phi(x) \arrow[r, "\phi(g)"] \arrow[d, "\sigma"'] & s.\phi(xg) \arrow[d, "\sigma"] \\ x \arrow[r, mapsto, "\phi(g)"'] & xg \end{tikzcd}$$

Simply transitive actions from reflection groups

One place where simply transitive actions arise is from tilings.

The group $\langle A, B \mid AB = BA \rangle \cong \mathbb{Z} \times \mathbb{Z}$ acts simply transitively on the unit squares in $\mathbb{Z}^2$.

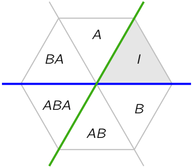




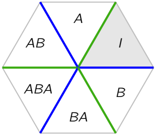
The shaded region is called a **fundamental chamber**.

Simply transitive actions from finite reflection groups

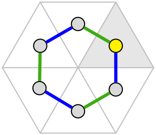
The dihedral group $D_3 = \langle A, B \mid A^2 = B^2 = (AB)^3 = 1 \rangle$ acts simply transitively on the six regions of a hexagon.



"*left action*"

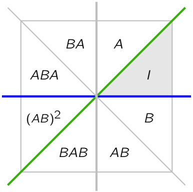


"*right action*"

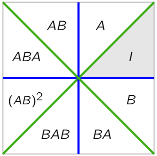


"*(right) Cayley graph*"

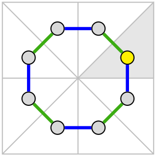
The dihedral group D_4 acts simply transitively on the eight regions of a square.



"*left action*"



"*right action*"

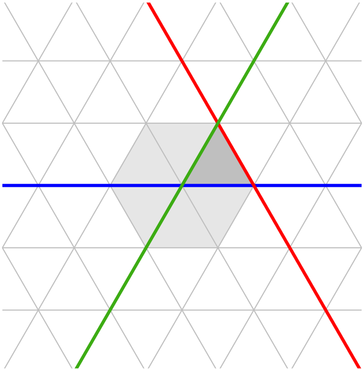


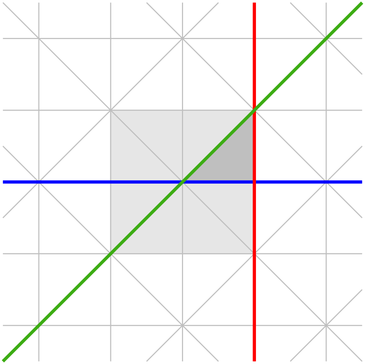
"*(right) Cayley graph*"

Simply transitive actions from affine reflection groups

In both previous examples, adding a third reflection generates a tiling of the plane.

The resulting affine groups, $\mathsf{Aff}(D_3)$ and $\mathsf{Aff}(D_4)$, act simply transitively on the chambers.

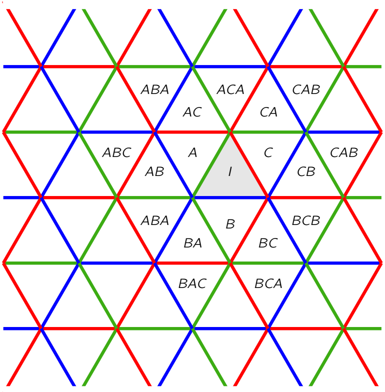


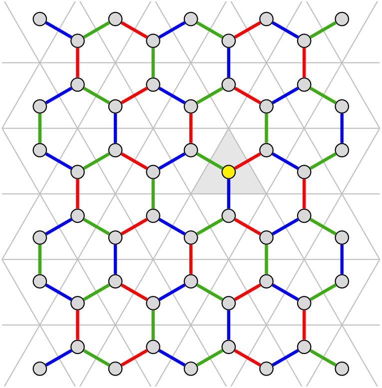


Simply transitive actions and affine Weyl groups

The group $\mathrm{Aff}(D_3)$ is better known as the affine Weyl group of type A_2.

It acts simply transitively on the chambers of the following tiling of $\mathbb{R}^2$.





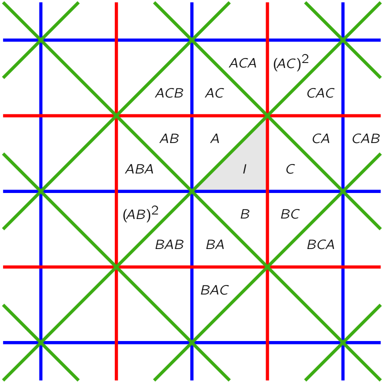
It has presentation

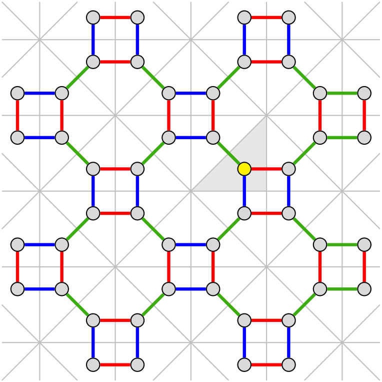
$$W(\tilde{A}_2) = \mathrm{Aff}(D_3) = \langle A, B, C \mid A^2 = B^2 = C^2 = (AB)^3 = (AC)^3 = (BC)^3 = 1 \rangle.$$

Simply transitive actions and affine Weyl groups

The group $\mathrm{Aff}(D_4)$ is better known as the affine Weyl group of type C_2.

It acts simply transitively on the chambers of the following tiling of $\mathbb{R}^2$.





It has presentation

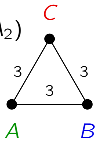
$$W(\tilde{C}_2) = \mathrm{Aff}(D_4) = \langle A, B, C \mid A^2 = B^2 = C^2 = (AB)^4 = (AC)^4 = (BC)^2 = 1 \rangle.$$

Weyl groups and Dynkin diagrams

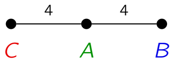
The presentations of the affine Weyl groups are encoded by Dynkin diagrams.

Nodes s_i are generators, and the labeled edges m_{ij} describe relations: $(s_i s_j)^{m_{ij}} = 1$.

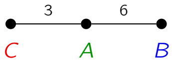
$\mathrm{Aff}(D_3) = W(\tilde{A}_2)$



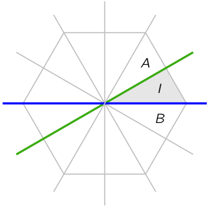
$\mathrm{Aff}(D_4) = W(\tilde{C}_2)$

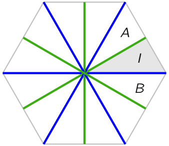


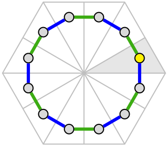
$\mathrm{Aff}(D_6) = W(\tilde{G}_2)$



This last example is the affine version of $D_6 = \langle A, B \mid A^2 = B^2 = (AB)^6 = 1 \rangle$ acting simply transitively on the 12 regions of a hexagon.



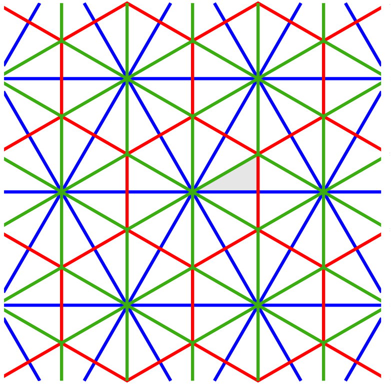


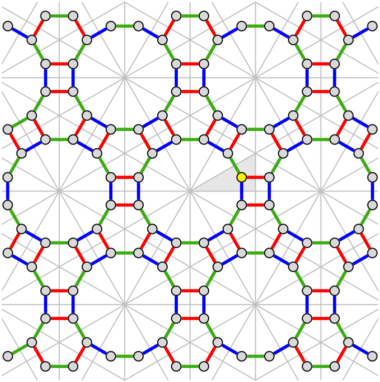


One last affine Weyl group

The group $\mathrm{Aff}(D_6)$ is better known as the the affine Weyl group of type G_2.

It acts simply transitively on the chambers of the following tiling of $\mathbb{R}^2$.





It has presentation

$$W(\tilde{G}_2) = \mathrm{Aff}(D_6) = \langle A, B, C \mid A^2 = B^2 = C^2 = (AB)^6 = (AC)^3 = (BC)^2 = 1 \rangle.$$

Coxeter groups and tilings of hyperbolic space

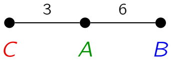
A Coxeter group is a group generated by "reflections", with presentation

$$W = \left\langle s_1, \dots, s_n \mid s_i^2 = 1,\ (s_i s_j)^{m_{ij}} = 1 \right\rangle.$$

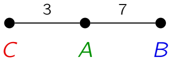
Like Weyl groups, this can be encoded by a Coxeter graph.

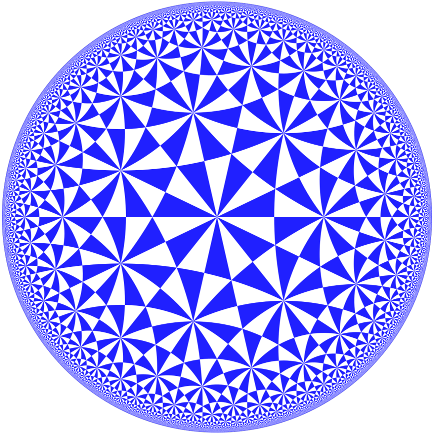
Some Coxeter groups act simply transitively on chambers of hyperbolic tilings.

$\mathrm{Aff}(D_6) = W(\tilde{G}_2)$



A hyperbolic Coxeter group





A simply transitive action of $\mathsf{PSL}_2(\mathbb{Z})$

The projective special linear group

$$\mathsf{PSL}_2(\mathbb{Z}) = \mathsf{SL}_2(\mathbb{Z})/\langle -I\rangle, \qquad \text{where } \mathsf{SL}_2(\mathbb{Z}) = \left\langle \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{S}, \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{T} \right\rangle$$

defines a tiling of hyperbolic ideal triangles in the upper half-plane via

$$S\colon z \longmapsto \frac{0z-1}{z+0} = -\frac{1}{z}, \qquad \text{and} \qquad T\colon z \longmapsto \frac{z+1}{0z+1} = z+1,$$