Visual Algebra

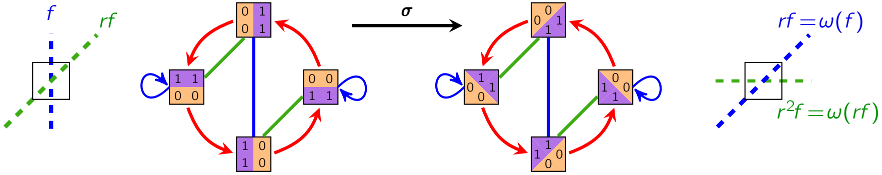
Lecture 5.9: Equivariance and G-set homomorphisms

Dr. Matthew Macauley

School of Mathematical & Statistical Sciences
Clemson University
South Carolina, USA
http://www.math.clemson.edu/~macaule/

Recall: Action equivalence (weaker) vs. G-set isomorphism (stronger)

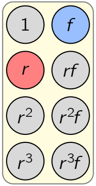
The following actions are equivalent, via $\omega \in \mathrm{Aut}(D_4)$, where $f \mapsto rf$,

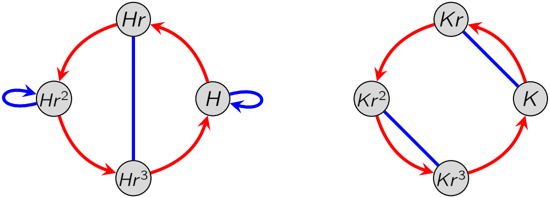


However, the corresponding G-sets are non-isomorphic.

$H = \mathbf{stab}\Big($  $\Big) = \langle f \rangle, \qquad K = \mathbf{stab}\Big($  $\Big) = \langle rf \rangle.$

"Switchboard"





Equivariant maps

Definition

Suppose G acts on S_i via $\phi_i \colon G \to \mathrm{Perm}(S_i)$ for $i = 1, 2$. A **G-equivariant map** is a function $\sigma \colon S_1 \to S_2$ such that $\sigma \circ \phi_1(g) = \phi_2(g) \circ \sigma$, for all $g \in G$:

$$
\begin{array}{ccc}
S_1 & \xmapsto{\phi_1(g)} & S_1 \\
\sigma \downarrow & & \downarrow \sigma \\
S_2 & \xmapsto{\phi_2(g)} & S_2
\end{array}
\qquad\qquad
\begin{array}{ccc}
s_1 & \xmapsto{\phi_1(g)} & s_1.\phi_1(g) \\
\sigma \downarrow & & \downarrow \sigma \\
s_2 & \xmapsto{\phi_2(g)} & s_2.\phi_2(g)
\end{array}
$$

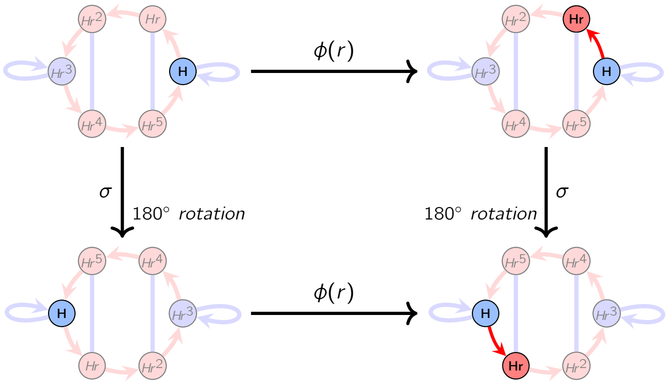
Loosely speaking, "*equivariance*" means "*commutes with the action*."

Key concepts

- Action equivalence involves an isomorphism $\iota \colon G_1 \to G_2$ and bijection $\sigma \colon S_1 \to S_2$.
- G-set isomorphisms ($\iota = \mathrm{Id}$, σ is bijective) are equivariant bijections.
- G-set automorphisms ($\iota = \mathrm{Id}$, σ is bijective, $S_1 = S_2$) form a group $\mathrm{Aut}_G(S)$.
- G-set homomorphisms ($\iota = \mathrm{Id}$, but σ need not be bijective) are equivariant maps.

G-set automorphisms as symmetries of the action graph

Let $S = G\backslash H$, for $G = D_6$ and $H = \langle f \rangle$.

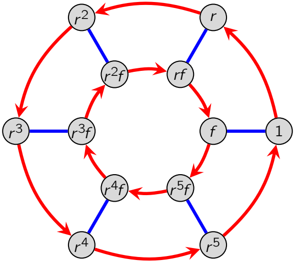


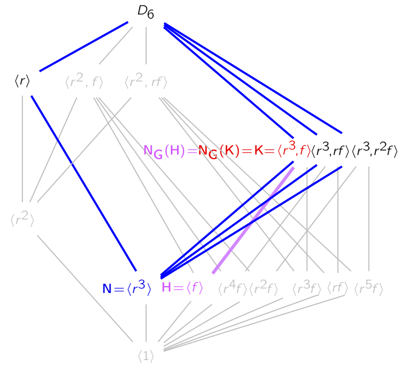
Key idea

The action and bijection clearly commute upon thinking of:

- the action $\phi(r)$ as right-multiplying Hr^i by r,
- the bijection σ as left-multiplying Hr^i by r^3. (This works because $r^3 \in N_G(H)$.)

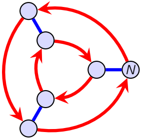
G-set automorphisms





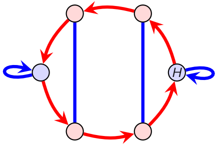
What do you notice about normalizers vs. symmetries of the actions graphs?

$N=\langle r^3\rangle$; *normal*



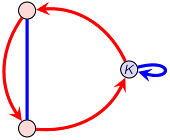
$\mathrm{Aut}_G(N\backslash G)\cong D_3\cong N_G(N)/N$

$H=\langle f\rangle$; *moderately unnormal*



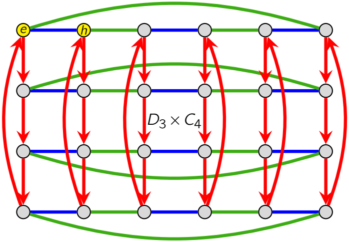
$\mathrm{Aut}_G(H\backslash G)\cong C_2\cong N_G(H)/H$

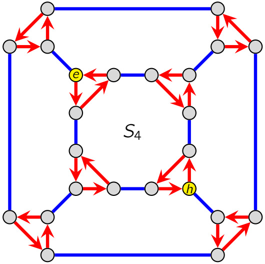
$K=\langle r^3,f\rangle$; *fully unnormal*

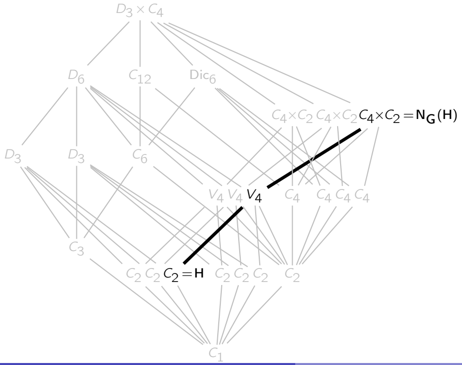


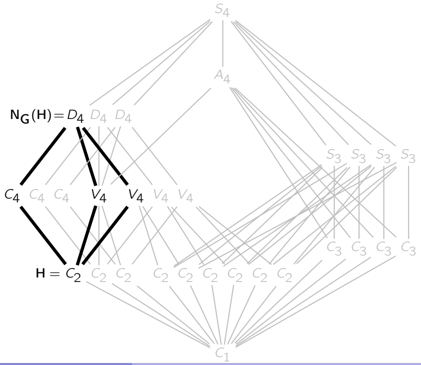
$\mathrm{Aut}_G(K\backslash G)\cong \langle 1\rangle\cong N_G(K)/K$

G-set automorphisms

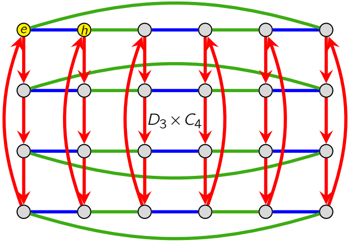


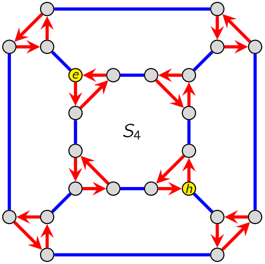






G-set automorphisms



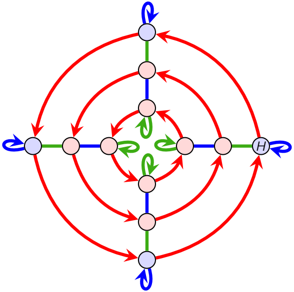


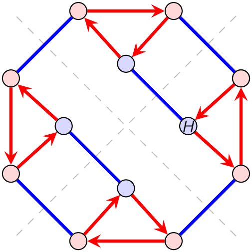
$H = \langle (f, 1) \rangle \leq D_3 \times C_4 = G$

$\mathrm{Aut}_G(H \backslash G) \cong N_G(H)/H \cong C_4$

$H = \langle ((12)(34)) \rangle \leq S_4 = G$

$\mathrm{Aut}_G(H \backslash G) \cong N_G(H)/H \cong V_4$





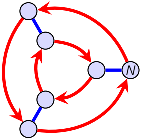
The G-set automorphism group

Theorem

For any $H \leq G$, the G-set automorphism group of $S = H\backslash G$ is

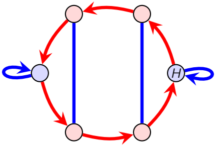
$$\mathrm{Aut}_G(S) \cong N_G(H)/H.$$

$N = \langle r^3 \rangle$; *normal*



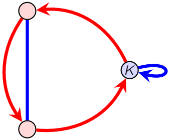
$\mathrm{Aut}_G(N\backslash G) \cong D_3$

$H = \langle f \rangle$; *moderately unnormal*



$\mathrm{Aut}_G(H\backslash G) \cong C_2$

$K = \langle r^3, f \rangle$; *fully unnormal*



$\mathrm{Aut}_G(K\backslash G) \cong C_1$

Here's how the proof will go, given $\sigma \in \mathrm{Aut}_G(S)$, and $S = H\backslash G$:

1. **Lemma 1**: $\sigma \colon Hg \mapsto Hxg$, for some fixed $x \in G$ (call this σ_x).
2. **Lemma 2**: $\sigma_x \in \mathrm{Aut}_G(S)$ iff $x \in N_G(H)$. That is, $\sigma_x \colon Hg \mapsto xHg$.
3. **FHT**: Two $\sigma_x = \sigma_{x'}$ iff x, x' are in the same coset of H.

The G-set automorphism group

Lemma 1

Any G-set automorphism $\sigma \in \mathrm{Aut}_G(S)$, for $S = H\backslash G$, is determined by the image of H:

$$\text{if } \sigma\colon H \mapsto Hx, \quad \text{then} \quad \sigma\colon Hg \mapsto Hxg, \text{ for all } g \in G.$$

Proof

Since σ is G-equivariant, it commutes with each $\phi(g) \in \mathrm{Perm}(S)$.

That is, the following diagram commutes:

$$\begin{array}{ccc} S & \xrightarrow{\ \phi(g)\ } & S \\ {\scriptstyle\sigma}\big\downarrow & & \big\downarrow{\scriptstyle\sigma} \\ S & \xrightarrow{\ \phi(g)\ } & S \end{array} \qquad\qquad \begin{array}{ccc} H & \xmapsto{\ \phi(g)\ } & Hg \\ {\scriptstyle\sigma}\big\downarrow & & \big\downarrow{\scriptstyle\sigma} \\ Hx & \xmapsto{\ \phi(g)\ } & Hxg \end{array}$$

It follows that $\sigma\colon Hg \mapsto Hxg$, as claimed. $\square$

The G-set automorphism group

Lemma 2

Let $S = H\backslash G$. The map of right cosets

$$\sigma_x \colon S \longrightarrow S, \qquad \sigma_x \colon Hg \longmapsto Hxg$$

is a G-set automorphism iff $x \in N_G(H)$.

Proof

"$\Rightarrow$": Suppose $\sigma_x \in \mathrm{Aut}_G(H\backslash G)$, and take $h \in H$. We have:

$$\begin{array}{ccc} S & \xrightarrow{\ \phi(h)\ } & S \\ {\scriptstyle \sigma_x}\downarrow & & \downarrow{\scriptstyle \sigma_x} \\ S & \xrightarrow{\ \phi(h)\ } & S \end{array} \qquad\qquad \begin{array}{ccc} H & \xmapsto{\ \phi(h)\ } & H \\ {\scriptstyle \sigma_x}\downarrow & & \downarrow{\scriptstyle \sigma_x} \\ Hx & \xmapsto{\ \phi(h)\ } & Hxh = Hx \end{array}$$

That is, for every $h \in H$,

$$H = Hxhx^{-1} \quad \Leftrightarrow \quad xhx^{-1} \in H \quad \Leftrightarrow \quad x \in N_G(H). \qquad \checkmark$$

The G-set automorphism group

Lemma 2

Let $S = H\backslash G$. The map of right cosets

$$\sigma_x \colon S \longrightarrow S, \qquad \sigma_x \colon Hg \longmapsto Hxg$$

is a G-set automorphism iff $x \in N_G(H)$.

Proof

"$\Leftarrow$": Suppose $x \in N_G(H)$, and pick $g \in G$.

By Lemma 1: $\sigma_x \colon Hg \mapsto Hxg = xHg$.

The operations σ_x (left-multiplying by x), and $\phi(g)$ (right-multiplying by g) clearly commute. ✓

$$\begin{array}{ccc} S & \xrightarrow{\phi(g)} & S \\ {\scriptstyle \sigma_x}\downarrow & & \downarrow{\scriptstyle \sigma_x} \\ S & \xrightarrow{\phi(g)} & S \end{array} \qquad \qquad \begin{array}{ccc} H & \overset{\phi(g)}{\longmapsto} & Hg \\ {\scriptstyle \phi(x)}\downarrow & & \downarrow{\scriptstyle \phi(x)} \\ Hx & \overset{\phi(g)}{\longmapsto} & Hxg = xHg \end{array}$$

The G-set automorphism group

Theorem

If G acts on the set $S = H\backslash G$ of right cosets of $H \leq G$, then

$$\mathsf{Aut}_G(S) \cong N_G(H)/H.$$

Proof

We'll apply the FHT to the map

$$f\colon N_G(H) \longrightarrow \mathsf{Aut}_G(S), \qquad x \longmapsto \sigma_x,$$

where $\sigma_x\colon Hg \longmapsto Hxg$.

Homomorphism: Straightforward exercise. ✓

Onto: Immediate from Lemma 2. ✓

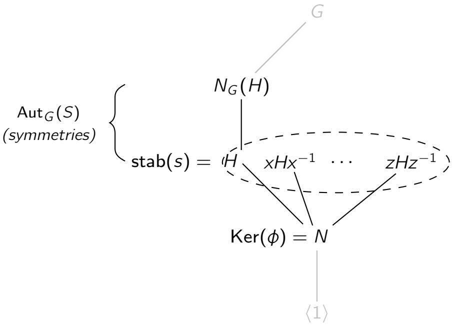
$\mathsf{Ker}(f) = H$. "$\subseteq$":

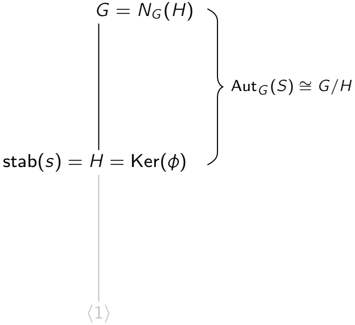
$$x \in \mathsf{Ker}(\phi) \quad \Leftrightarrow \quad Hg = Hxg,\ \forall\, g \in G \quad \Leftrightarrow \quad H = Hx \quad \Leftrightarrow \quad x \in H.$$

"$\supseteq$": If $h \in H$, then $\sigma_h\colon Hg \mapsto Hhg = Hx$. ✓

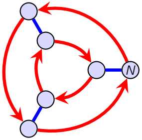
The result now follows from the FHT. □

G-set automorphisms



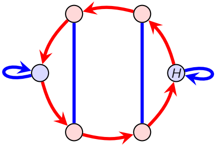


$N = \langle r^3 \rangle$; *normal*



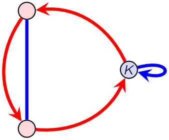
$\text{Aut}_G(N\backslash G) \cong D_3$

$H = \langle f \rangle$; *moderately unnormal*



$\text{Aut}_G(H\backslash G) \cong C_2$

$K = \langle r^3, f \rangle$; *fully unnormal*

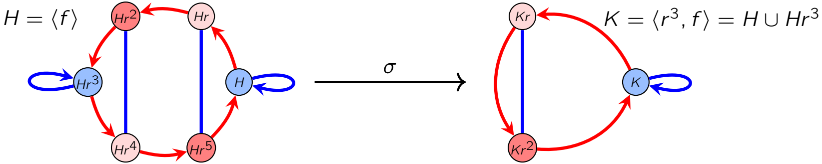


$\text{Aut}_G(K\backslash G) \cong C_1$

G-set homomorphisms

Dropping bijectivity of $\sigma \colon S_1 \to S_2$ defines a G-set homomorphism, or G-equivariant map.

Consider this example of D_6-sets:



This can be described by the following commutative diagram:

$$\begin{array}{ccc} H\backslash G & \xmapsto{\phi(g)} & H\backslash G \\ \sigma \downarrow & & \downarrow \sigma \\ K\backslash G & \xmapsto{\phi(g)} & K\backslash G \end{array} \qquad \begin{array}{ccc} Hx & \xmapsto{\phi(g)} & Hxg \\ \sigma \downarrow & & \downarrow \sigma \\ Kx & \xmapsto{\phi(g)} & Kxg \end{array}$$

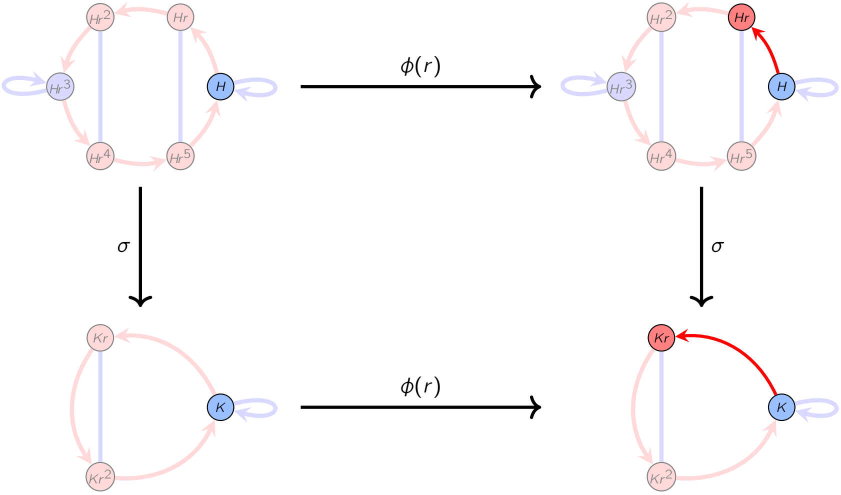
Key idea

We say that "*the map σ commutes with the action of the group*."

G-set homomorphisms

Here is that example again for $G = D_6$ and subgroups:

$H = \langle f \rangle$ (*moderately unnormal*), $\quad K = \langle r^3, f \rangle = H \cup Hr^3 = N_G(H)$, (*fully unnormal*)



The "fundamental homomorphism theorem for G-sets"

Orbit-stabilizer theorem, restated

If $\phi\colon G \to \mathsf{Perm}(S)$ is a transitive action and $s \in S$, then $\mathsf{orb}(s)$ is isomorphic to the quotient of G by $H = \mathsf{stab}(s)$:

$$H \setminus G \cong \mathsf{orb}(s).$$