# Visual Algebra

## Lecture 8.10: Rings of fractions

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# Motivation: constructing $\mathbb{Q}$ from $\mathbb{Z}$

Rational numbers are ordered pairs under an equivalence, e.g., $\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \cdots$

## Equivalence of fractions

Given $a, b, c, d \in \mathbb{Z}$, with $b, d \neq 0$,

$$\frac{a}{b} = \frac{c}{d} \qquad \text{if and only if} \qquad ad = bc.$$

We can mimic this construction in any integral domain.

## Definition

Given an integral domain $R$, its field of fractions is the set

$$R \times R^* = \{(a, b) \mid a, b \in R,\ b \neq 0\},$$

under the equivalence $(a_1, b_1) \sim (a_2, b_2)$ iff $a_1 b_2 = b_2 a_1$.

Denote the class containing $(a, b)$ as $a/b$. Addition and multiplication are defined as

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \qquad \text{and} \qquad \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

It's not hard to show that $+$ and $\times$ are well-defined.

# Embedding integral domains in fields

## Lemma

In the construction of the field of fractions from $R$, we must verify:

- $\sim$ is a equivalence relation
- the $+$ and $\times$ operations are well-defined on $(R \times R^*)/\sim$
- the additive identity is $0/r$ for any $r \in R^*$
- the multiplicative identity is $r/r$ for any $r \in R^*$
- $(a, b)^{-1} = b/a$.

| **Integral domain** | **Field of fractions** |
|---|---|
| $\mathbb{Z}$ (integers) | $\mathbb{Q}$ (rationals) |
| $\mathbb{Z}[i]$ (Gaussian integers) | $\mathbb{Q}(i)$ (Gaussian rationals) |
| $F[x]$ (polynomials) | $F(x)$ (rational functions) |

Every integral domain canonically embeds into its field of fractions, via $r \mapsto r/1$.

Moreover, this is the *minimal* field containing $R$.

# Co-universal property of the field of fractions

## Proposition

Let $R$ be an integral domain with embedding $\iota\colon R \hookrightarrow F_R$ into its field of fractions. Then for every other embedding $f\colon R \hookrightarrow K$ into a field, there is a unique $h\colon F_R \hookrightarrow K$ such that $h \circ \iota = f$.

$$
\begin{tikzcd}
R \arrow[rr, hook, "f"] \arrow[dr, hook, "\iota"'] & & K \\
& F_R \arrow[ur, hook, dashed, "\exists ! h"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
r \arrow[rr, mapsto, "f"] \arrow[dr, mapsto, "\iota"'] & & f(r) \\
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\end{tikzcd}
$$

## Proof

Define the map

$$h\colon F_R \longrightarrow K, \qquad h(a/b) \longmapsto h(a/1)h(b/1)^{-1} = f(a)f(b)^{-1}.$$

We need to show that $h$ is

(i) well-defined

(ii) a ring homomorphism,

(iii) injective

(iv) unique.

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**(i) Well-defined.** Suppose $\quad a/b = c/d \quad \Leftrightarrow \quad ad = bc \quad \Leftrightarrow \quad ab^{-1} = cd^{-1}$.

$$h(a/b) = f(a)f(b^{-1}) = f(ab^{-1}) = f(cd^{-1}) = f(c)f(d^{-1}) = h(c/d). \qquad \checkmark$$

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## Proof

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$$h\colon F_R \longrightarrow K, \qquad h(a/b) \longmapsto h(a/1)h(b/1)^{-1} = f(a)f(b)^{-1} = f(a)f(b^{-1}).$$

**(ii) Ring homomorphism**. Suppose $a/b = c/d$. Then

$$\begin{aligned} h(a/b \cdot c/d) = h(ac/bd) &= f(ac)f(d^{-1}b^{-1}) = f(a)f(c)f(d^{-1})f(b^{-1}) \\ &= f(a)f(b^{-1}) \cdot f(c)f(d^{-1}) = h(a/b)h(c/d). \end{aligned} \qquad \checkmark$$

Verification of $h(a/b + c/d) = h(a/b) + h(c/d)$ is similar. (Exercise)

# Co-universal property of the field of fractions

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## Proof

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**(iii) Injective**. It suffices to show that $\mathrm{Ker}(h) = \{0\}$. Suppose

$$0 = h(a/b) = f(a)f(b)^{-1} \in K.$$

However, $f(b)^{-1} \neq 0$ since $f$ is an embedding and $b \neq 0$.

Thus $f(a) = 0$, so $a = 0$ in $R$. Thus $a/1 = 0/1$, the zero element in $F_R$. ✓

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**(iv) Uniqueness**. Suppose there is another $g\colon F_R \to K$ such that $f = g \circ \iota$. Then

$$g(a/b) = g((a/1)\cdot(b/1)^{-1}) = g(a/1)g(b/1)^{-1} = g(\iota(a))g(\iota(b))^{-1} = f(a)f(b)^{-1} = h(a/b),$$

which completes the proof. $\square$

# Rings of fractions and localization

The co-universal property can be used as the *definition* of the field of fractions, allowing:

- the generalization to rings without 1, e.g., $R = 2\mathbb{Z}$. (Exercise: show that $F_{2\mathbb{Z}} = \mathbb{Q}$.)
- the generalization to constructing fractions of certain subsets.

Let $R$ be commutative, $D \subseteq R$ nonempty and multiplicatively closed with no zero divisors.

We can carry out the same construction of the set

$$R \times D = \big\{(r,d) \mid r \in R,\ d \in D\big\}, \qquad (r_1,d_1) \sim (r_2,d_2) \text{ iff } r_1 d_2 = r_2 d_1.$$

The resulting ring is the localization of $R$ at $D$, denoted $D^{-1}R$.

## Proposition (HW)

Let $R$ be a commutative ring with embedding $\iota\colon R \hookrightarrow D^{-1}R$. Then for every other embedding $f\colon R \hookrightarrow S$ to a ring where $f(D)$ are units, there is a unique $h\colon D^{-1}R \hookrightarrow S$ such that $h \circ \iota = f$.

$$\begin{tikzcd} R \arrow[rr, hook, "f"] \arrow[dr, hook, "\iota"'] & & S \\ & D^{-1}R \arrow[ur, hook, dashed, "\exists! h"'] & \end{tikzcd} \qquad \begin{tikzcd} r \arrow[rr, mapsto, "f"] \arrow[dr, mapsto, "\iota"'] & & f(r) \\ & r/1 \arrow[ur, mapsto, dashed, "h"'] & \end{tikzcd}$$

# Localization with zero divisors

We can generalize this further! Allow $D$ to contain zero divisors.

The mapping $R \to D^{-1}R$ sending $r$ to its equivalence class is no longer injective:

$$\iota\colon R \longrightarrow D^{-1}R, \qquad \iota(z) = 0, \quad \text{for all zero divisors } z \in D.$$

We still have a co-universal property, that could have been the definition.

## Proposition (exercise)

Let $R$ be a commutative ring with $\iota\colon R \to D^{-1}R$. For every other $f\colon R \to S$ to a ring where the non zero-divisors in $f(D)$ are units, there is a unique $h\colon D^{-1}R \to S$ such that $h \circ \iota = f$.

$$\begin{tikzcd} R \arrow[rr, "f"] \arrow[dr, "\iota"'] & & S \\ & D^{-1}R \arrow[ur, dashed, "\exists! h"'] & \end{tikzcd} \qquad \begin{tikzcd} r \arrow[rr, mapsto, "f"] \arrow[dr, mapsto, "\iota"'] & & f(r) \\ & r/1 \arrow[ur, dashed, mapsto, "h"'] & \end{tikzcd}$$

Thus, $D^{-1}R$ is the "smallest ring" where all non zero-divisors in $D$ are invertible.

# Examples of rings of fractions

1. If $R$ is an integral domain and $D = R^*$, then $D^{-1}R$ is its field of fractions.
2. If $D$ is the set of non zero divisors, then $D^{-1}R$ is the total ring of fractions of $R$.
3. If non-unit of $R$ is a zero divisor, then $R$ is equal to its total ring of fractions.

   Examples include $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$.

   In these rings, every prime ideal is maximal (exercise).
4. The localization of $R = F[x]$ at $D = \{x^n \mid n \in \mathbb{Z}\}$ are the Laurent polynomials:
$$D^{-1}R = F[x, x^{-1}] = \left\{a_{-m}x^{-m} + \cdots + a_{-1}x^{-1} + a_0 + a_1x + \cdots + a_nx^n \mid a_i \in F\right\}.$$
5. If $R = \mathbb{Z}$ and $D = \left\{5^n \mid n \in \mathbb{N}\right\}$, then
$$D^{-1}R = \mathbb{Z}[\tfrac{1}{5}] = \left\{a_0 + \frac{a_1}{5} + \frac{a_2}{5^2} + \cdots + \frac{a_n}{5^n} \mid a_i \in \mathbb{Z}\right\}.$$
   which are "*polynomials in* $\frac{1}{5}$" over $\mathbb{Z}$.
6. If $D = R - P$ for a prime ideal, then $R_P := D^{-1}R$ is the localization of $R$ at $P$. It is a local ring – it has a unique maximal ideal, $PR_P$.