Question. Chapter 6.1 and 6.2 of Strang (total of 20 marks)

1. Compute the eigenvalues and eigenvectors of the following matrices (6 marks)
   
   (a) \( A_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix} \)  
   
   (b) \( A_2 = A_1^{-1} \)  
   
   (c) \( A_3 = A_1^2 + 3I \)

2. Prove that if \( A \) is an invertible matrix and \( \lambda \) is an eigenvalue of \( A \), then \( \lambda^{-1} \) is an eigenvalue of \( A^{-1} \). (3 marks)

3. Prove that \( A \) is a diagonal matrix if and only if the standard basis vectors are all eigenvectors of \( A \). (3 marks)

4. Diagonalize matrix \( A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \) by finding the matrices \( S \) and \( \Lambda \). (4 marks)

5. Diagonalize \( A \) and compute \( SA^kS^{-1} \) to prove this formula for \( A^k \) (4 marks)

\[
A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad \text{and} \quad A^k = \frac{1}{2} \begin{pmatrix} 1 + 3^k & 1 - 3^k \\ 1 - 3^k & 1 + 3^k \end{pmatrix}
\]