MATH 8510 - Fall 2014

Homework 1

Due: Thursday, September 4

1. Answer the following questions about groups
   (a) Let G be the group of rigid motions of a cube. Determine the order of G.
   (b) Let G be a group such that for all $g \in G$, $g^2 = e$. Prove that G is Abelian.
   (c) Show every group of even order has an element of order 2.

2. Is it true that a semigroup with a left identity and in which every element has a right inverse is a group?

3. Answer the following questions about subgroups
   (a) Exhibit a proper subgroup of $\mathbb{Q}$ which is not cyclic.
   (b) Let H and K be subgroups of G. Then HK is a subgroup of G iff $KH = HK$.

4. Let G be a group and H a subgroup of G. H is a *normal* subgroup if it holds

$$\forall\, g \in G,\ h \in H, \quad ghg^{-1} \in H \text{ (or equivalently } gHg^{-1} \subseteq H).$$

   Answer the following questions
   (a) Prove that for any $g \in G$, gHg^{-1} is a subgroup of G and $|H| = |gHg^{-1}|$.
   (b) Let H be a finite group of order n. Prove that if H is the only subgroup of G with this order, then H is normal.
   (c) Let G be a group and H a subgroup of G. Prove that $\bigcap_{g \in G} gHg^{-1}$ is the largest normal subgroup of G contained in H.

5. The following problem proves Cauchy's theorem. Let G be a group and p a prime dividing $|G|$. Let $\mathcal{S}$ denote the set of p-tuples of elements of G the product of whose coordinates is 1:

$$\mathcal{S} = \left\{ (x_1, \cdots, x_p) : x_i \in G, \prod x_i = 1 \right\}.$$

   (a) Show that $\mathcal{S}$ has $|G|^{p-1}$ elements, hence has order divisible by p.

   Define the relation $\sim$ on $\mathcal{S}$ by letting $\alpha \sim \beta$ if β is a cyclic permutation of α.

   (b) Show that a cyclic permutation of an element of $\mathcal{S}$ is again an element of $\mathcal{S}$.
   (c) Prove that $\sim$ is an equivalence relation on $\mathcal{S}$.
   (d) Prove that an equivalence class contains a single element if and only if it is of the form $(x, x, \cdots, x)$ with $x^p = 1$.
   (e) Prove that every equivalence class has order 1 or p. Deduce that $|G|^{p-1} = k + pd$, where k is the number of classes of size 1 and d is the number of classes of size p.
   (f) Since $\{(1, \cdots, 1)\}$ is an equivalence class of size 1, conclude that there must be a nonidentity element x in G with $x^p = 1$, i.e., G contains an element of order p.