MATH 8510 - Fall 2014

Homework 2

Due: Thursday, September 11

1. Answer the following questions about homomorphisms.
   (a) Let G and G' be groups and $\varphi : G \to G'$ be a homomorphism. Let $g \in G$. Prove that $|\varphi(g)| \big| |g|$.
   (b) Let G be a group and consider the map $\varphi : G \to G$ given by $\varphi(g) = g^2$. Prove that φ is a homomorphism iff G is abelian.
2. Answer the following questions about group actions.
   (a) Let G be a nonabelian group, then $g \cdot a = ag$ does not satisfy the conditions of a left action of G on itself.
   (b) Let G be a group of odd order and $g \in G$ be a nonidentity element. Prove that g is not conjugate to g^{-1}.
3. Answer the following questions about normal subgroups.
   (a) What group consists of the symmetries of a tetrahedron? Express your solution in terms of a group action.
   (b) What group consists of the rigid motions of a cube? Express your solution in terms of a group action.
4. Let p and q be primes with $p < q$. Let G be a group of order pq. Prove the following (without using the Sylow theorems):
   (a) G has subgroups of order p and q.
   (b) The subgroup of order q is normal in G.
   (c) If G is nonabelian, then there exists a nonnormal subgroup of order p.
5. Answer the following questions about group actions.
   (a) Let G be a group, H a subgroup, and N a normal subgroup. Assume that $|H|$ and $[G : N]$ are finite and relatively prime. Prove that H is a subgroup of N.
   (b) Let M, N be normal subgroups of G, prove that $NM/M \equiv N/N \cap M$.
   (c) Let M, N be normal subgroups of G such that $M \cap N = \{e\}$. Prove that $mn = nm$ for all $m \in M$ and $n \in N$.
6. Let G be a finite abelian group in which the number of solutions in G of the equation $x^n = e$ is at most n for every positive integer n. Prove that G must be cyclic.