MATH 8510 - Fall 2014

Homework 3

Due: Thursday, September 18

1. Let H and K be subgroups of a group G so that $K \subseteq H$ and $[G:H]$ and $[H:K]$ are finite. Prove that $[G:K]$ is finite and $[G:K] = [G:H][H:K]$.
2. Let G be the group $\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and let N be the cyclic subgroup $\langle(1,1)\rangle$. Describe the quotient group G/N.
3. Let G be a group and let A be the subset of G consisting of elements of the form $xyx^{-1}y^{-1}$. Let $[G,G]$ denote the subgroup of G generated by A. This subgroup is referred to as the commutator subgroup of G.
   (a) Prove that $[G,G]$ is normal in G.
   (b) Prove that $G/[G,G]$ is abelian.
4. Define a map $\phi : \mathbb{Z} \to \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ by $n \mapsto (n+3\mathbb{Z}, n+6\mathbb{Z})$. Prove that ϕ is a homomorphism. Is ϕ surjective? What is the kernel of ϕ?
5. Let p be a prime. Prove that $a^{p-1} \equiv 1 \pmod{p}$ for all $a \in \mathbb{Z}$ with $\gcd(a,p) = 1$.
6. Let G be a group. Let K be a subgroup of G and let $K\backslash G$ denote the set of right cosets.
   (a) If $g \in G$, show that the map $\phi_g : K\backslash G \to K\backslash G$ given by $\phi_g(Kb) = Kbg$ is a permutation of the set $K\backslash G$.
   (b) Prove that the function $\psi : G \to Sym(K\backslash G)$ given by $\psi(g) = \phi_{g^{-1}}$ is a homomorphism of groups with kernel contained in K.
   (c) If K is normal in G, prove that $K = \ker(\psi)$.
   (d) Use these results to prove Cayley's theorem, namely, that every group is isomorphic to a group of permutations.
7. Let $N \in \mathbb{Z}_{>1}$ and define
$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\}.$$
   (a) Prove that $\Gamma_0(N)$ is a subgroup of $SL_2(\mathbb{Z})$.
   (b) Prove that $\Gamma_1(N)$ is a normal subgroup of $\Gamma_0(N)$ where
$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \mid a \equiv d \equiv 1 \pmod{N}. \right\}$$
   (c) Describe the quotient group $\Gamma_0(N)/\Gamma_1(N)$.