MATH 8510 - Fall 2014

Homework 4

Due: Thursday, September 25

1. Let G be the group
$$\mathrm{GL}_2^+(\mathbb{R}) = \{g \in \mathrm{GL}_2(\mathbb{R}) : \det(g) > 0\}.$$
Let $\mathcal{H}$ denote the complex upper half-plane, i.e.,
$$\mathcal{H} = \{z = x + iy \in \mathbb{C} \mid y > 0\}.$$
   (a) Show that G acts on $\mathcal{H}$ via $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$. (Don't forget to show this actually maps into $\mathcal{H}$ in addition to showing it is a group action.)

   (b) Compute $\mathrm{Stab}_G(i)$.

2. Let p and q be distinct primes so that $q \not\equiv 1 \pmod{p}$ and $p \not\equiv 1 \pmod{q}$. If G is a group of order p^2q, show that G is isomorphic to $\mathbb{Z}/p^2q\mathbb{Z}$ or $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$.

3. Let G be a finite group with $|G| > 1$.
   (a) Prove that G has a composition series, i.e., there exists a sequence of groups

   (b) Assume G has two composition series
$$\{e\} = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_n = G$$
$$\{e\} = M_0 \trianglelefteq M_1 \trianglelefteq M_2 = G.$$
   Show that $n = 2$.

   (c) Let
$$\{e\} = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_n = G$$
$$\{e\} = M_0 \trianglelefteq M_1 \trianglelefteq \cdots \trianglelefteq M_m = G$$
   be two composition series for G. Use induction on $\min(m, n)$ and part (b) to prove that $m = n$ and there is a permutation σ so that $M_i/M_{i-1} \sim N_{\sigma(i)}/N_{\sigma(i)-1}$ for all i. (Hint: Apply the induction hypothesis to $H = N_{n-1} \bigcap M_{m-1}$.)

   (d) Give a composition series for $\mathbb{Z}/15\mathbb{Z}$.

   (e) Give an example of an infinite group G and subgroups $\{N_i\}$ so that
$$G = N_0 \trianglerighteq N_1 \trianglerighteq N_2 \trianglerighteq \cdots$$
   that does not terminate, i.e., $N_m \neq \{e\}$ for all m.

4. Let G be a finite group, p a prime, and $p^k \mid |G|$ for some $k \geq 0$. Modify the proof of Sylow's Theorem to show that G has a subgroup of order p^k .

5. Let p be an odd prime and G a group of order $2p$. This problem shows that G is isomorphic to $\mathbb{Z}/2p\mathbb{Z}$ or D_p.
   (a) Prove that G has an element a of order p and an element b of order 2.

   (b) Prove that there is a t so that $bab = a^t$. Furthermore, show that $a^{t^2} = a$. From this, conclude that $t \equiv \pm 1 \pmod{p}$.

   (c) Prove that G is isomorphic to $\mathbb{Z}/2p\mathbb{Z}$ or D_p.

6. Let $G = \mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})$.
   (a) Find $|G|$.

   (b) Give all 3-Sylow subgroups of G.

   (c) Prove that the subgroup of G generated by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ is the unique 2-Sylow subgroup of G.

   (d) Show that $Z(G) = \pm 1_2$ where $1_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Use this to show that $G/Z(G) \sim A_4$.