MATH 8510 - Fall 2014

Homework 7

Due: Thursday, October 30

1. Let K be a field. A *discrete valuation* on K is a function $\nu : K^* \longrightarrow \mathbb{Z}$ satisfying

   - ν is a homomorphism from the multiplicative group K^* to the additive group $\mathbb{Z}$
   - ν is surjective
   - $\nu(x+y) \geq \min\{\nu(x), \nu(y)\}$ for all $x, y \in K^*$ with $x + y \neq 0$.

   The set $R = \{x \in K^* : \nu(x) \geq 0\} \cup \{0\}$ is called the *valuation ring* of ν.

   (a) Prove that R is a subring of K that contains the identity. (A ring R is called a *valuation ring* if there is a field K and a discrete valuation ν so that R is the valuation ring of ν.)

   (b) Prove that for each $x \in K^*$, either x or x^{-1} is in R.

   (c) Prove that $x \in R^*$ if and only if $\nu(x) = 0$.

   (d) Let $K = \mathbb{Q}$ and let p be a prime in $\mathbb{Z}$. Define $\nu_p : \mathbb{Q}^* \to \mathbb{Z}$ by $\nu_p\left(\frac{a}{b}\right) = r$ where $\frac{a}{b} = p^r \frac{c}{d}$ with $p \nmid cd$. Check that ν_p is a valuation. Describe the valuation ring and the units of the valuation ring.

2. Let R be a ring and G a group.

   (a) The *center* of a ring R is the set

   $$Z(R) = \{z \in R : zr = rz \text{ for all } r \in R\}.$$

   Prove the center is a subring of R.

   (b) If $G = \{g_1, \ldots, g_n\}$, prove that the element $g_1 + \cdots + g_n$ is in the center of the group ring $R[G]$.

   (c) Let $R = \mathbb{Z}$ and $G = S_3$. Let $\alpha = 3(1\,2) - 5(2\,3) + 14(1\,2\,3)$ and $\beta = 6(1) + 2(2\,3) - 7(1\,3\,2)$ be elements of $R[G]$. Compute $2\alpha - 3\beta$, $\alpha\beta$, and α^2.

3. Let R and S be nonzero ring. Let $\varphi : R \to S$ be a nonzero homomorphism of rings.

   (a) Prove that if $\varphi(1) \neq 1$, then $\varphi(1)$ is a zero divisor in S. Deduce that if S is an integral domain then every ring homomorphism from R to S sends the identity of R to the identity of S.

   (b) Prove that if $\varphi(1) = 1$ then $\varphi(u)$ is a unit in S and that $\varphi(u^{-1}) = \varphi(u)^{-1}$ for each $u \in R^*$.

4. Let $\varphi : R \to S$ be a homomorphism of commutative rings.

   (a) Prove that if $\mathfrak{p}$ is a prime ideal of S, then either $\varphi^{-1}(\mathfrak{p}) = R$ or $\varphi^{-1}(\mathfrak{p}) \in \operatorname{Spec}(R)$. Apply this to the special case when R is a subring of S and φ the inclusion map to conclude that if $\mathfrak{p}$ is a prime ideal of S, then $\mathfrak{p} \cap R$ is either R or in $\operatorname{Spec}(R)$.

   (b) Prove that if $\mathfrak{m} \in \text{M-Spec}(S)$ and φ is surjective, then $\varphi^{-1}(\mathfrak{m}) \in \text{M-Spec}(R)$. Is this still true if one removes the assumption that φ is surjective? If it is true, prove it. If not, give a counter-example.

5. Let R be a commutative ring.

   (a) We say an element $x \in R$ is *nilpotent* if $x^n = 0$ for some $n \in \mathbb{Z}_{>0}$. The set of nilpotent elements is denoted $\mathfrak{N}(R)$ and called the *nilradical* of R. Prove the nilradical is an ideal.

   (b) Let I be an ideal in R. Define the *radical of I* to be

   $$\operatorname{rad}(I) = \{x \in R : x^n \in I \text{ for some } n \in \mathbb{Z}_{>0}\}.$$

   Prove that $\operatorname{rad}(I)$ is an ideal containing I and $(\operatorname{rad}(I))/I$ is the nilradical of the quotient ring R/I.

(c) Let $\mathrm{Jac}(I)$ be the intersection of all maximal ideals in R containing I. Prove this is an ideal containing I. This ideal is known as the Jacobson radical of I.

(d) Prove that $\mathrm{rad}(I) \subseteq \mathrm{Jac}(I)$.

(e) Let $n \in \mathbb{Z}_{>1}$. Describe $\mathrm{Jac}(n\mathbb{Z})$ in terms of the prime factorization of n.

6. Let $x^2 + x + 1 \in R = \mathbb{F}_2[x]$ and use bar notation to denote passage to the quotient ring $R/(x^2 + x + 1)$.

   (a) Prove that $\overline{R}$ has 4 elements.

   (b) Write out the addition table for $\overline{R}$ and deduce that the additive group $\overline{R}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

   (c) Write out the multiplication table for $\overline{R}$ and prove that $\overline{R}^*$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$. Deduce that $\overline{R}$ is a field.

7. A proper ideal $\mathcal{Q}$ of a commutative ring R is called *primary* if whenever $xy \in \mathcal{Q}$ and $x \notin \mathcal{Q}$ then $y^n \in \mathcal{Q}$ for some $n \in \mathbb{Z}_{>0}$. Prove the following facts about primary ideals.

   (a) The primary ideals of $\mathbb{Z}$ are (0) and $p^n\mathbb{Z}$ where p is prime and $n \in \mathbb{Z}_{>0}$.

   (b) Every prime ideal of R is a primary ideal.

   (c) An ideal $\mathcal{Q}$ of R is primary if and only if every zero divisor in $R/\mathcal{Q}$ is in $\mathfrak{N}(R/\mathcal{Q})$.

   (d) If $\mathcal{Q}$ is a primary ideal then $\mathrm{rad}(\mathcal{Q}) \in \mathrm{Spec}(R)$.