MATH 8530 - Fall 2015

Homework 1

Due: Sep. 22 (Tuesday)

Question 1.

Prove that the set $\{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$ together with the natural operations of addition and multiplication is a field.

Question 2.

Let X be a vector space over a field $\mathbb{F}$. Let 0 be the zero element of $\mathbb{F}$ and 0 the zero-element of X. Using only the definitions of a group, a vector space, and a field, carefully prove each of the following:

1. The identity element e of a group is unique.
2. In any group G, the inverse of $g \in G$ is unique.
3. $0x = 0$ for every $x \in X$;
4. $k0 = 0$ for every $k \in \mathbb{F}$;
5. For every $k \in \mathbb{F}$ and $x \in X$, if $kx = 0$, then $k = 0$ or $x = 0$.

Question 3.

Let Y be a subspace of X a vector space over a field $\mathbb{F}$ and consider the quotient space $X/Y = \{[x] \mid x \in X\}$. Prover that the operations

$$[x_1] + [x_2] = [x_1 + x_2] \quad \text{and} \quad \lambda[x] = [\lambda x]$$

where $x, x_1, x_2 \in X$ and $\lambda \in \mathbb{F}$ are well-defined, meaning that they do not depend on the choice of congruent class representative.

Question 4.

Let S be a set of vectors in a finite-dimensional vector space X. Show that S is a basis of X if every vector of X can be written in one and only one way as a linear combination of the vectors in S.

Question 5.

Let Y_1, Y_2 be subspace of a vector space X over a field $\mathbb{F}$ such that $Y_1 \cap Y_2 = \{0\}$. Prove that $Y_1 + Y_2$ is isomorphic to $Y_1 \times Y_2$.