MATH 8530 - Fall 2015
Homework 5

Due: Nov. 3 (Tuesday)

QUESTION 1.

Fix a matrix $A \in \mathbb{R}^{3\times3}$. We define a linear operator $T : \mathbb{R}^{3\times3} \to \mathbb{R}^{3\times3}$ by $T(B) = AB$ for all $B \in \mathbb{R}^{3\times3}$. Find the characteristic polynomial of the map T.

QUESTION 2.

1. Let $A \in \mathbb{F}^{n\times n}$. Prove that $A \in \mathrm{GL}_n(\mathbb{F})$ if and only if 0 is not an eigenvalue of A.
2. Let $\lambda \in \mathbb{F}$ be and eigenvalue of $A \in \mathrm{GL}_n(\mathbb{F})$. Prove that λ^{-1} is an eigenvalue of A^{-1}.

QUESTION 3.

1. Let $P(x) = \sum_{i=1}^{d} P_i x^i,\ Q(x) = \sum_{j=1}^{t} Q_j x^j \in \mathbb{F}^{n\times n}[x]$ and $R(x) = P(x)Q(x)$. Prove that for a matrix $A \in \mathbb{F}^{n\times n}$ satisfying $Q_j A = AQ_j$ for all $j = 1, \dots, t$ it holds $R(A) = P(A)Q(A)$.
2. For $n = 2$, find an example of $P(x)$ and $Q(x)$ such that $R(B) \neq P(B)Q(B)$ for some nn matrix B.

QUESTION 4.

Let A be an $n \times n$ matrix over $\mathbb{C}$ with distinct eigenvalues $\lambda_1, \dots, \lambda_n$. For a vector $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, define the *norm* of z by

$$||z|| = \Big(\sum_{i=1}^{n} |z_i|\Big)^{1/2}.$$

1. Prove that if $|\lambda_i| < 1$ for all i, then $||A^N z|| \to 0$ as $N \to \infty$ for all $z \in \mathbb{C}^n$.
2. Prove that if $|\lambda_i| > 1$ for all i, then $||A^N z|| \to \infty$ as $N \to \infty$ for all nonzero $z \in \mathbb{C}^n$.

QUESTION 5.

Let A be an $n \times n$ matrix over $\mathbb{C}$ with an eigenvalue λ of index $m \geq 2$ and corrresponding eigenvector v_1. Let v_2 be a generalized eigenvector satisfying $(A - \lambda I)v_2 = v_1$.

1. Prove that for any natural number N it holds $A^N v_2 = \lambda^N v_2 + N\lambda^{N-1} v_1$.

2. Prove that for any polynomial $q(x) \in \mathbb{C}[x]$,

$$q(A)v_2 = q(\lambda)v_2 + q'(\lambda)v_1 \,,$$

   where $q'(x)$ is the derivative of q.

3. Conjecture a formula for $q(A)v_m$, where $v_1, \ldots, v_m$ are generalized eigenvectors of A with $(A - \lambda I)v_k = v_{k-1}$ (and say $v_0 = 0$, for convenience).