MATH 8530 - Fall 2015
Homework 6

Due: Nov. 10 (Tuesday)

QUESTION 1.

Let X be an n-dimensional vector space, and $A\colon X \to X$ a linear map with distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. Let $v_1, \ldots, v_n$ be the corresponding eigenvectors of A, and let $\ell_1, \ldots, \ell_n$ be the corresponding eigenvectors of the transpose $A'\colon X' \to X'$.

1. Prove that $(\ell_i, v_i) \neq 0$ for $i = 1, \ldots, n$.
2. Show that if $x = a_1 v_1 + \cdots + a_n v_n$, then $a_i = (\ell_i, x)/(\ell_i, v_i)$.
3. Is $\ell_1, \ldots, \ell_n$ necessarily the dual basis of $v_1, \ldots, v_n$? Why or why not?

QUESTION 2.

Compute the Jordan canonical form and provide the matrix of change of basis for

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ -4 & -2 & -1 & -2 \\ -2 & -3 & 2 & -4 \end{pmatrix}$$

QUESTION 3.

et A be a matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. The *index* of λ_i is the minimum integer d_i such that $(A - \lambda_i I)^{d_i} = 0$ but $(A - \lambda_i I)^{d_i - 1} \neq 0$.

1. Prove, using the spectral theorm but without appealing to Jordan canonical form, that the minimal polynomial of A is
$$m_A(s) = \prod_{i=1}^{k} (s - \lambda_i)^{d_i} .$$
2. Give a simple proof using the Jordan canonical form.

QUESTION 4.

Find a list of real matrices, as long as possible, such that:

1. The characteristic polynomial of each matrix is $p(s) = (s-1)^5(s+1)$.
2. The minimal polynomial of each matrix is $m(s) = (s-1)^2(s+1)$.
3. No two matrices in the list are similar to each other.

QUESTION 5.

Let $X \subset \mathbb{R}[x, y]$ be the space of polynomials in x, y of total degree $\leq n$. Find the dimension of X, show that the map

$$A : X \longrightarrow X, \qquad f \longmapsto f + \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}$$

is linear, and find its Jordan canonical form. [*Hint*: First find all eigenvectors.]