Homework 7

QUESTION 1.

Let f and g be continuous functions on the interval $[0, 1]$. Prove the following inequalities.

1. $\left(\int_0^1 f(t)g(t)\, dt \right)^2 \leq \int_0^1 f(t)^2\, dt \int_0^1 g(t)^2 dt$

2. $\left(\int_0^1 (f(t) + g(t))^2\, dt \right)^{1/2} \leq \left(\int_0^1 f(t)^2\, dt \right)^{1/2} + \left(\int_0^1 g(t)^2 dt \right)^{1/2}.$

QUESTION 2.

Let X be the vector space of all continuous real-valued functions on $[0, 1]$. Define an inner product on X by

$$(f, g) = \int_0^1 f(t)g(t)\, dt\,.$$

Let Y be the subspace of X spanned by f_0, f_1, f_2, f_3, where $f_k(x) = x^k$. Find an orthonormal basis for Y.

QUESTION 3.

Let Y be a subspace of a Euclidean space X, and $P_Y \colon X \to X$ the orthogonal projection onto Y. Prove that $P_Y^* = P_Y$.

QUESTION 4.

Show that a matrix M is orthogonal if and only if its column vectors form an orthonormal set.