MATH 8560 - Spring 2014

Homework 1

Due: Jan. 28th (Tuesday)

Question 1. **(2 marks)**

(Alternative proof of the chain rule for entropy)
Let X_1, X_2 and X_n be random variables such that $(X_1, X_2, \ldots, X_n) \sim p(x_1, \ldots, x_n)$. Prove that

$$p(x_1, \ldots, x_n) = \prod_{i=1}^{n} p(x_i \mid x_{i-1}, \ldots, x_1)).$$

Show also that the chain rule for entropy, i.e.,

$$H(X_1, \ldots, X_n) = \sum_{i=1}^{n} H(X_i \mid X_{i-1}, \ldots, X_1),$$

is a consequence of the previous formulation of the joint probability.

Question 2. **(1 marks)**

(From Jensen's Inequality)
Let f be a strictly convex function and X a discrete random variable. Prove that if $E[f(X)] = f(E[X])$, then $X = E[X]$ with probability 1, i.e., X is a constant.

Question 3. **(4 marks)**

(Coin flips)
A fair coin is flipped until the first head occurs. Let X denote the number of flips required.

1. Find the entropy $H(X)$ in bits. The following expressions my be useful:

$$\sum_{i=0}^{\infty} r^i = (1-r)^{-1} \text{ and } \sum_{i=0}^{\infty} i r^i = r(1-r)^{-2}.$$

2. A random variable X is drawn according to this distribution. Find an "efficient" sequence of yes-no questions of the form "Is X contained in the set S?". Compare $H(X)$ to the expected number of questions required to determine X.

Question 4. **(2 marks)**

(Zero conditional entropy)
Show that is $H(Y \mid X) = 0$, then Y is a function of X, i.e., for all x such that $p(x) > 0$, there exists only one possible value of y such that $p(x, y) > 0$.

Question 5. **(6 marks)**

(Entropy of a sum)
Let X and Y be random variables that take on values $x_1, x_2, \ldots, x_r$ and $y_1, y_2, \ldots, y_s$, respectively. Let $Z = X + Y$.

1. Show that $H(Z \mid X) = H(Y \mid X)$. Argue that if X, Y are *independent*, then $H(Y) \leq H(Z)$ and $H(X) \leq H(Z)$. Thus, the addition of independent random variables add uncertainty.
2. Give an example of random variables in which $H(X) > H(Z)$ and $H(Y) > H(Z)$.
3. Under what conditions does $H(Z) = H(X) + H(Y)$?