MATH 8560 - Spring 2014

Homework 2

Due: Feb. 13th (Thursday)

Question 1. **(2 marks)**

(AEP and mutual information)

Let (X_i, Y_i) be i.i.d. $\sim p(x, y)$. We form the likelihood ration of the hypothesis that X and Y are dependent vs. the one that they are independent. What is the limit of

$$\frac{1}{n} \log \frac{p(X^n)p(Y^n)}{p(X^n, Y^n)}?$$

Question 2. **(3 marks)**

(High probability sets)

For each $n \in$, let $B_\delta^{(n)} \subset \mathcal{X}^n$ to be the smallest set with $\Pr\{B_\delta^{(n)}\} \geq 1 - \delta$. Prove the following statements:

1. $\Pr(A_\epsilon^{(n)} \cap B_\delta^{(n)}) \geq 1 - \epsilon - \delta$, and
2. for sufficiently large n, there exists a $\delta' > 0$ such that $|B_\delta^{(n)}| \geq (1 - \delta')2^{n(H-\epsilon)}$

Question 3. **(2 marks)**

(Additive noise channel)

Find the channel capacity of the following DMC:

$$Y = X + Z$$

where $\Pr\{Z = 0\} = \Pr\{Z = a\} = 1/2$ and the alphabet for X is $\{0, 1\}$. Assume that Z is independent of X.

Question 4. **(4 marks)**

(Errors and Erasures in a binary channel)

Find the capacity of the channel $(\mathcal{X}, p(y|x), \mathcal{Y})$ with $\mathcal{X} = \{0, 1\}$, $\mathcal{Y} = \{0, e, 1\}$ and

$$p(y|x) = \begin{cases} \alpha & \text{if } y = e \\ \epsilon, & \text{if } x \neq y \neq e \\ 1 - \alpha - \epsilon, & \text{if } x = y. \end{cases}$$

Question 5. **(4 marks)**

(Noise alphabet)

Consider the alphabets $\mathcal{X} = \{0, 1, 2, 3\}$ and $\mathcal{Z} = \{z_1, z_2, z_3\}$. Let Z be a uniformly distributed random variable on $\mathcal{Z}$. Consider the channel $Y = X + Z$.

1. What is the maximum capacity over all the choices of $\mathcal{Z}$? Find a realization of this capacity.
2. What is the minimum capacity over all the choices of $\mathcal{Z}$? Find a realization of this capacity.