MATH 8560 - Spring 2016

Homework 1

Due: Jan. 28th (Thursday)

QUESTION 1. *From Jensen's Inequality.*

Let f be a strictly convex function and X a discrete random variable. Prove that if $E[f(X)] = f(E[X])$, then $X = E[X]$ with probability 1, i.e., X is a constant.

QUESTION 2. *Coin flips.*

A fair coin is flipped until the first head occurs. Let X denote the number of flips required.

1. Find the entropy $H(X)$ in bits. The following expressions my be useful:

$$\sum_{i=0}^{\infty} r^i = (1-r)^{-1} \text{ and } \sum_{i=0}^{\infty} i r^i = r(1-r)^{-2}.$$

2. A random variable X is drawn according to this distribution. Find an "efficient" sequence of yes-no questions of the form "Is X contained in the set S?". Compare $H(X)$ to the expected number of questions required to determine X.

QUESTION 3. *Zero conditional entropy.*

Show that is $H(Y \mid X) = 0$, then Y is a function of X, i.e., for all x such that $p(x) > 0$, there exists only one possible value of y such that $p(x, y) > 0$.

QUESTION 4. *Capacity.*

Compute the capacities of the following channels:

1. *Z-channel*: $\mathcal{X} = \mathcal{Y} = \{0, 1\}$ and

$$p(y \mid x) = \left(\begin{array}{c|cc} x \setminus y & 0 & 1 \\ \hline 0 & 1 & 0 \\ 1 & p & 1-p \end{array} \right)$$

2. *Symmetric channel*: $\mathcal{X} = \mathcal{Y} = \{0, 1, 2\}$ and

$$p(y \mid x) = \left(\begin{array}{c|ccc} x \setminus y & 0 & 1 & 2 \\ \hline 0 & 1/2 & 1/8 & 3/8 \\ 1 & 3/8 & 1/2 & 1/8 \\ 2 & 1/8 & 3/8 & 1/2 \end{array} \right)$$

3. *(Errors and Erasures in a binary channel)*: $\mathcal{X} = \{0, 1\}$, $\mathcal{Y} = \{0, e, 1\}$ and

$$p(y|x) = \begin{cases} \alpha & \text{if } y = e \\ \epsilon, & \text{if } x \neq y \neq e \\ 1 - \alpha - \epsilon, & \text{if } x = y. \end{cases}$$