QUESTION 1. From Jensen’s Inequality.

Let $f$ be a strictly convex function and $X$ a discrete random variable. Prove that if $E[f(X)] = f(E[X])$, then $X = E[X]$ with probability 1, i.e., $X$ is a constant.

QUESTION 2. Coin flips.

A fair coin is flipped until the first head occurs. Let $X$ denote the number of flips required.

1. Find the entropy $H(X)$ in bits. The following expressions may be useful:

$$\sum_{i=0}^{\infty} r^i = (1 - r)^{-1} \quad \text{and} \quad \sum_{i=0}^{\infty} ir^i = r(1 - r)^{-2}.$$

2. A random variable $X$ is drawn according to this distribution. Find an “efficient” sequence of yes-no questions of the form “Is $X$ contained in the set $S$?”. Compare $H(X)$ to the expected number of questions required to determine $X$.


Show that if $H(Y \mid X) = 0$, then $Y$ is a function of $X$, i.e., for all $x$ such that $p(x) > 0$, there exists only one possible value of $y$ such that $p(x, y) > 0$.


Compute the capacities of the following channels:

1. Z-channel: $\mathcal{X} = \mathcal{Y} = \{0, 1\}$ and 

$$p(y \mid x) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 1 & p \end{pmatrix}$$

2. Symmetric channel: $\mathcal{X} = \mathcal{Y} = \{0, 1, 2\}$ and 

$$p(y \mid x) = \begin{pmatrix} 0 & 1/2 & 1/8 & 3/8 \\ 1 & 3/8 & 1/2 & 1/8 \\ 2 & 1/8 & 3/8 & 1/2 \end{pmatrix}$$

3. (Errors and Erasures in a binary channel): $\mathcal{X} = \{0, 1\}, \mathcal{Y} = \{0, e, 1\}$ and 

$$p(y \mid x) = \begin{cases} \alpha, & \text{if } y = e \\ \epsilon, & \text{if } x \neq y \neq e \\ 1 - \alpha - \epsilon, & \text{if } x = y. \end{cases}$$