QUESTION 1. Log likelihood ratio.

Let \( S = (\mathbb{F}_2, p(y \mid x), \mathcal{Y}) \) be a binary memoryless channel such that \( p(y \mid x) \geq 0 \) for any \( y \in \mathcal{Y} \) and \( x \in \mathbb{F}_2 \). The log likelihood ratio for a \( y \in \mathcal{Y} \) is

\[
\mu(y) := \log_2 \left( \frac{p(y \mid 0)}{p(y \mid 1)} \right).
\]

Let \( C \) be a \((n, M)\) code over \( \mathbb{F}_2 \), and define the decoder \( D : \mathcal{Y}^n \to C \) such that for any \( y = (y_1, \ldots, y_n) \in \mathcal{Y}^n \)

\[
D(y) = \arg\max_{c \in C} \sum_{i=1}^{n} (-1)^{c_i} \mu(y_i).
\]

Show that \( D \) is a maximum-likelihood decoder for \( C \) with respect to \( S \).

QUESTION 2. Decoding failure

Show that for every \((n, M, d)\) code \( C \) over \( X \) and for every decoder \( D : X^n \to C \), there is a codeword \( c \in C \) and a vector \( y \in X^n \) such that \( d(y, c) \leq \lfloor (d + 1)/2 \rfloor \) and \( D(y) \neq c \).

QUESTION 3. Some properties of linear codes

Let \( C \subset \mathbb{F}_2^n \) be a \([n, k] \) linear code.

1. Show that either any codeword has even weight, or exactly half of them have even weight.
2. If \( C \) has a codeword of odd weight, then show that the even weight codewords of \( C \) form an \([n, k - 1] \) linear code.
3. Show that either all codewords in \( C \) begin with zero, or exactly half of them begin with zero.
4. Show that the sum of the weights of all codewords in \( C \) is at most \( n2^{k-1} \).

QUESTION 4. Puncturing a linear code

Let \( C \) be a linear \([n, k, d] \) code over a field \( F \). For \( i = 1, \ldots, n \) denotes with \( C_i \) the code

\[
C_i := \{(c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_n) \mid (c_1, \ldots, c_n) \in F^n \}.
\]

The code \( C_i \) is said to be obtained by puncturing \( C \) at the \( i \)-th coordinate.

- Show that \( C_i \) is a linear \([n - 1, k_i, d_i] \) code over \( F \) where \( k_i \geq k - 1 \) and \( d_i \geq d - 1 \).
- Show that there are at least \( n - k \) indices \( i \) for which \( k_i = k \).